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CROSS-COUPLING
IN
MULTIPLE BEAM ANTENNAS

Scientific Report No. 1
on
AF19(604)-7237

APRIL 1961

Prepared for
ELECTRONICS RESEARCH DIRECTORATE
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS

SYLVANIA ELECTRONIC SYSTEMS
Government Systems Management
for GENERAL TELEPHONE & ELECTRONICS

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"CROSS-COUPLING IN MULTIPLE BEAM ANTENNAS"

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THEORY OF ANTENNA PERFORMANCE

IN SCATTER-TYPE RECEPTION

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ABSTRACT

This report comprises two memoranda, each with its own abstract. The reader is referred to these.

PREFATORY NOTE

In the course of research under this contract on methods for more efficiently utilizing antennas in fading radio communications, such as typified by tropospheric scatter, it was desired for comparison purposes to carry out detailed performance estimates for angle diversity systems. Such systems have been proposed on the basis of experimental observations that the apparent angle of arrival of such fading signals tends to wander over several beamwidths at the receiving point; it then appears economical in equipment to consider a diversity system based on the use of a single large reflector with a number of feeds arranged to produce squinted beams. Each feed is connected in such a system to a separate receiver, and the outputs of the receivers are combined in the appropriate diversity combining circuitry, for example, so-called maximal-ratio combining.

A number of analyses of angle diversity have been reported to date.[1-6]

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1. S. Stein, D.E. Johansen, and A.W. Starr, "Theory of Antenna Performance in Scatter-Type Reception". Hermes Electronics Co. Report M783, AFCRC-TR-59-191, 30 Sept. 1959 (Appendix F).

2. A.B. Crawford, D.C. Hogg, and W.H. Kummer, "Studies in Tropospheric Propagation Beyond the Horizon", BSTJ, 38, pp. 1067-1178 (1959).

3. J.H. Vogelman, J.L. Ryerson, and M.H. Bickelhaupt, "Tropospheric Scatter System Using Angle Diversity", Proc. IRE, 47, pp. 688-696 (1959).

4. R. Bolgiano, Jr., N.H. Bryant, and W.E. Gordon, "Diversity Reception in Scatter Communications with Emphasis on Angle Diversity", Cornell Univ., Final Report, Part I, on AF30(602)-1717, Jan. 1958.

5. C. Chu, A.H. Wren, and J. LaRue, "Evaluation of the Pincushion System", Univ. Mich. Rad. Lab. Rept. 2872-1-T, RADC-TN-60-50, Feb. 1960.

6. W.R. Richard and M.H. Bickelhaupt, "Multiple Angle Diversity Design Considerations", Rome Air Development Center, RADC-TN-60-22, April 1960.

Problems which are discussed in this scientific report were first noted in an earlier analysis[1] where it was disturbing to find that casual angle diversity calculations led to results which were obviously inconsistent with conservation of energy. The same type of mathematics appears, or seems to be implied, in all the other discussions of angle diversity.[2-6] The problem which arises is the following: The "casual diversity calculation", referred to above, is one which assumes that each of the various beams of the receiving array collects energy from the incident field, essentially independently of the presence of the other beams. That is, one calculates the received power in each beam from a knowledge of the incident field, or equivalently of the sources creating the field, by applying the reciprocity theorem with the radiation pattern of each beam only, assuming that there are no interaction effects due to the presence of the other beams which would diminish the estimate of received power. The basis for this assumption is that the beams do not greatly overlap. However, although for reasons of physical construction, standard multiple feed microwave antenna systems usually cannot produce beams which cross over at levels higher than about their 3 db points, it is not at all clear that with some ingenuity useful beams could not be designed which would overlap even more greatly. Surprisingly, the analysis in this report indicates that even for beams crossing over at their 3 db points the modifications indicated by the analysis can be quite significant.

Of course, somewhat paradoxically, all the angle diversity analyses have shown an awareness of the need for taking account of beam overlap insofar as estimating the degree of correlation of the fluctuations in the energies received on the various beams, in the case of fading propagation channels typified by tropospheric transhorizon ("scatter") systems. However, the issue to be discussed in this report is, in fact, quite separate from the fluctuating or nonfluctuating nature of the incident field. It is rather a statement of inherent limitation in the character of the multiple beam antenna systems, due to the necessity of the conservation of energy principle. In short, the relations to be described here refer to a limitation on the energy which can be collected at each instant by the antenna system in a fading environment, and do not refer at all to time fluctuations of such collected energy.

The nature of the problem under discussion becomes obvious by considering as an example the extreme limiting case of completely overlapping beams. For example, suppose one has an antenna system with one feed, but chooses to put M different taps on the feedline at various points along the circuit. Looking at everything between those taps and the radiation patterns as a "black box antenna system", which one is certainly free to do (if only to be different), gives an antenna system with M "feed-lines" entering the "black box", each exciting a perfectly good radiation pattern when excited, with all the other taps terminated in matched loads (which is, indeed, how one defines the beam patterns in any multiple beam system). Thus, we would observe that our black box is a multiple-beam antenna system which has M identical, completely overlapping beams. It is certainly clear on the one hand that one cannot buy something for nothing, that in reality there exists only the <u>one</u> radiation pattern; that if in receiving, the outputs of the taps are combined in an optimum linear combining network, the net results will <u>at best</u> be the same as if one had initially adopted the more conventional attitude that there exists only the one feed, and therefore, only one real beam collecting energy and one terminal point. In fact, of course, if one excites one of the taps in our example, some fraction of the energy will disappear into the loads associated with the other taps and only the remaining fraction into the radiated beam. In other words, the presence of these other taps, or alternatively the fact that the beams overlap as much as they do, creates an insertion loss between each tap and its associated beam. It is this insertion loss, which appears in reception as well as in transmission, which is inherent in the principle of conservation of energy, and for which we seek a characterization.

In the extreme example just cited, the problem was obvious and the correction factors equally obvious. The challenge which appeared, however, since we wanted to analyze angle diversity configurations in which the beams crossed over at varying squint angles, and hence with varying degrees of overlap, was to produce a mathematical formulation and description of such a system which will automatically and routinely account correctly for these limitations. Such a formulation would then be completely dependable for performance predictions for angle diversity systems, regardless of how peculiar the feeding system (or the way in which the inventor chooses to regard the system).

However, this mathematical formulation appeared possibly to be of much more general significance to designers of multiple beam antenna systems. For this reason, while the analysis performed under this contract achieved its goal once a formulation giving the correct results for angle diversity had been achieved, it seemed desirable to further explore these results in the more general context. Although this further exploration was carried out external to the contract, for the sake of completeness the entire contents are being reported in this scientific report, with reference to the more general context.

This scientific report consists of two engineering memoranda which are entitled:

Part I: Radiation Efficiencies (with Application to Angle Diversity)

Part II: Characteristics of the Cross-Coupling Matrices

The contents of each are perhaps obvious from the titles. It is our belief, as stated already, that a significant new insight has been obtained through this work into many of the problems of designing multiple beam antenna systems. Nevertheless, as will be apparent after reading this report, much further useful work remains to be done if the full significance of these results is to be obtained with respect to particular system applications. These conclusions are reiterated perhaps more meaningfully at the ends of the two memoranda.

ON CROSS-COUPLING IN MULTIPLE BEAM ANTENNAS

PART I RADIATION EFFICIENCIES

(With Applications to Angle Diversity)

by

S. Stein

ABSTRACT

A general multiple-beam antenna system is considered, in which the beam patterns overlap. The degree of overlap is defined by a cross-correlation type of integral which includes phase factors as well as polar diagrams in the description of the beams. It is shown that when beam overlap exists, conservation of energy implies the existence of unavoidable cross-couplings between the feed-lines, and a related limitation on the radiation efficiency (or corresponding receiving cross section) of any single beam when excited alone. Certain problems relating to the angle diversity concept are discussed, by way of example. In addition, some unsolved corollary problems with possibly important design applications are outlined.

In Part II (a companion paper), the scattering matrix comprised of the array of cross-coupling factors between the various feed-lines is considered. Some properties are derived, and implications discussed. In addition, further unsolved problems relating to characterization of the scattering matrix are also outlined.

1. Introduction:

There are many multiple-beam antenna systems in use or contemplated in which the radiating structure may be regarded as driven by a number of feed-lines, each feed-line corresponding to excitation of a particular beam. The various beams may be excited simultaneously or sequentially, the assumption being made below that the radiating structure is not changed physically simply because a particular feed-line is not excited. When there is overlap among any of the beams which the structure is designed to radiate (overlap in a sense to be defined below), it becomes apparent that the corresponding excited portions of the radiating structure are likely to overlap, and therefore that cross-coupling may exist between the corresponding feed-lines. In fact, it will be shown below that such beam overlap renders cross-coupling unavoidable, and it is the object of the paper to describe various implications in terms of the beam patterns. Most of the attention in this Part I is given to deriving the basic equations, and subsequently discussing radiation efficiency problems, with particular attention to the angle diversity concept. In Part II[1], the detailed characterization of the cross-couplings is treated.

2. Scattering Matrix Representation of the Antenna System.

Consider an antenna system comprising M feed-lines, feeding into a radiating structure which we regard as a junction. We define (Fig. 1) the k^{th} beam pattern as that beam radiated when a generator is exciting the k^{th} feed-line, and it and all other feed-lines are terminated in matched loads which absorb any energy reflected back from the junction.

Rather than describe the antenna system in terms of voltages and currents, or impedances, it will be useful to employ an incident and reflected wave representation (similar to the description of transmission lines in terms of forward- and backward-traveling waves). Specifically, let us consider a steady-state, single

[1] S. Stein and J.E. Storer, "On Cross-Coupling In Multiple Beam Antennas, Part II. Characteristics of the Cross-Coupling Matrices," ARM-238, 2 March 1961.

frequency situation (harmonic time dependence). We will denote the amplitudes of the waves traveling towards the junction in each of the feed-lines by the set of complex-valued numbers $\{x_k\}$, k = 1, ..., M and the waves traveling away from the junction at the same time by the set $\{y_k\}$, k = 1, ..., M. We define the scales of the individual x_k and y_k in each line such that a unit magnitude corresponds to a unit power (1 watt) level in the wave (power transfer by the wave across a cross-section of the line). The phase factors of the x_k and y_k refer to some arbitrary reference point in each line.

We further assume the antenna system contains no non-linear elements. Each of the y_k is therefore related to the x_k by some linear sum, of the form

$$y_k = \sum_{m=1}^{M} S_{km} x_m \tag{1}$$

where the S_{km} are so-called scattering coefficients. If we write the arrays of the x_k and y_k as single-column matrices (vectors),

$$x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_M \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_M \end{pmatrix} \tag{2}$$

and denote the M x M array of the S_{km} as the scattering matrix, S (S_{km} being the element in the k^{th} row and m^{th} column), we can rewrite Eq. 1 as the matrix equation

$$y = Sx \tag{3}$$

Next, let us define the M beams in detail as follows: Consider a generator in the k^{th} feed-line, exciting a unit amplitude wave incident on the radiating structure in the k^{th} feed-line, and with all other feed-lines terminated in matched loads so that there is no mechanism to produce incident waves in these other feed-lines, i.e.,

$$x_k = 1$$
$$x_j = 0 , \quad j \neq k$$

Under these circumstances, let the complex-valued field radiated by the antenna structure be described (in the far-zone) by

$$\vec{E}_k(\vec{\Theta}) = q_k \vec{R}_k(\vec{\Theta}) \frac{\exp(i \frac{2\pi}{\lambda} r)}{r} \tag{4}$$

The vector $\vec{\Theta}$ represents the spatial angle coordinates, and r the radial distance from a reference origin located at the antenna structure. The explicit $\exp(i \frac{2\pi}{\lambda} r)$ dependence on r is the usual outgoing spherical wave form describing the far-zone field, with the form of the angular dependence described by $\vec{R}_k(\vec{\Theta})$. The vector form for $\vec{R}_k$ and $\vec{E}_k$ denotes the possibility of two crossed linear polarization components (e.g., "horizontal" and "vertical") at each point in the far-zone. (Each polarization component is also complex-valued.) We label the angular dependence $\vec{R}_k(\vec{\Theta})$ as the <u>k^{th} beam pattern</u>. We will assume $\vec{R}_k(\vec{\Theta})$ to be conveniently normalized such that

$$\frac{\eta_o}{2} \int \vec{R}_k^*(\vec{\Theta}) \cdot \vec{R}_k(\vec{\Theta}) \, d\vec{\Theta} = 1 \tag{5}$$

where $\eta_o = \frac{1}{120\pi}$ is the "admittance of free-space"; the asterisk denotes a complex conjugate; the usual vector dot product (scalar product) is indicated; and the integration, as indicated symbolically, is over all spatial angles. With this normalization, the magnitude of the coefficient, q_k, is the relative amplitude of the field, and its phase refers the phase of the field to that of the wave incident in the k^{th} feed-line and to the definition of a phase reference in $\vec{R}_k(\vec{\Theta})$. In particular, the total radiated power in the far-zone field of Eq. (4) is given by

$$P_k = \int r^2 \frac{\eta_o}{2} \vec{E}_k^*(\vec{\Theta}) \cdot \vec{E}_k(\vec{\Theta}) \, d\vec{\Theta} \tag{6}$$

Using Eq. (4), and the normalization in Eq. (5), this reduces to

$$P_k = q_k^* q_k = \left|q_k\right|^2 \tag{7}$$

where bars indicate the magnitude of complex quantities. Since unit power is incident upon the junction, $\left|q_k\right|^2$ may be considered to represent the radiation efficiency for this k^{th} beam. Also, with the particular excitation and loading specified, $1 - \left|q_k\right|^2$ must represent energy carried away from the junction in reflected waves in the several feed-lines and absorbed in their terminating matched loads; alternatively described, $\left|q_k\right|^2$ is an insertion loss between the input wave on the k^{th} feed-line, and the radiated k^{th} beam.

3. Beam-Coupling Factors

Having defined the M beam patterns, we now define a measure of the beam overlaps. The appropriate measure, as will be seen later, is the set of beam-coupling factors defined by

$$\beta_{kj} = \frac{\eta_o}{2} \int \vec{R}_k^*(\vec{\Theta}) \cdot \vec{R}_j(\vec{\Theta}) \, d\vec{\Theta} \tag{8}$$

Clearly,

$$\beta_{kj} = \beta_{jk}^* \quad \text{for all } k, j \tag{9a}$$

and by Eq. (5),

$$\beta_{kk} = 1 \tag{9b}$$

$$\left|\beta_{kj}\right|^2 \leq 1 \tag{9c}$$

It is important to note that Eq. (8) does <u>not</u> define beam coupling simply in terms of overlap of the usual polar diagrams. The fact that the $\vec{R}_k(\vec{\theta})$ are complex-valued can be extremely important, since rapid variations in phase in the product being integrated can cause the integral to vanish. Such rapid variations can occur, for example, if the locations on the radiating structure of the major currents exciting the two beams are sufficiently disjoint. For instance, if a set of currents gives rise to a beam $\vec{R}(\vec{\theta})$, and this set of currents is translated by a distance d, the new pattern ($\vec{\theta}$ and r retaining their original meaning) will have the form

$$\exp\left[-\,i2\pi\,\frac{d}{\lambda}\cos\psi\right]\vec{R}(\vec{\theta})$$

where ψ is the angle between the direction of the observing point in space, and the direction in which the currents were translated. As another example, if we (loosely) use the Fourier Transform relationship between the beam pattern and the aperture plane fields for a microwave "aperture-type" antenna, the integral in Eq. (8) is readily shown to be equivalent to an integral

$$\int \vec{F}_k(\vec{s}) \cdot \vec{F}_j(\vec{s})\, d\vec{s}$$

where $\vec{s}$ represents coordinates in the aperture plane, and $\vec{F}(\vec{s})$ the aforementioned aperture fields. But then, if $\vec{F}_k(\vec{s})$ and $\vec{F}_j(\vec{s})$ are respectively non-vanishing only over disjoint sections of the aperture plane, the integral vanishes. In actual fact, such disjointness may be more a design ideal than a realized fact, due to induced currents, "leakages", etc; and many designs, e.g., phased arrays, of course do not have such disjointness to begin with. (In fact, beam pattern measurements may represent a more effective engineering approach to determining the actual extent of beam-couplings than trying to determine what actually are the totality of currents associated with any given beam pattern.)

Any pair of beams for which the corresponding beam coupling factor vanishes will be said to be decoupled.

4. Conservation of Energy Equations

We can now proceed to the main results of this paper. Consider again the steady-state situation envisaged in Section 2, with now a general excitation of the feed-lines such that waves are incident on the radiating structure in all feed-lines, with the set of amplitudes given by the column matrix, x. As described by Eq. (3), the associated reflected waves in all feeding lines are given by the column matrix, $y = Sx$. Clearly, however, we can now complete the picture: Corresponding to the incident wave of complex amplitude x_k in the k^{th} line, there is a radiated field,

$$\vec{E}_k(\vec{\Theta}) = x_k\, q_k\, \vec{R}_k(\vec{\Theta})\, \frac{\exp(i\,\frac{2\pi}{\lambda}\,r)}{r}$$

and hence, by superposition, the total radiated field associated with x is

$$\vec{E}(\vec{\Theta}) = \left[\sum_{k=1}^{M} x_k\, q_k\, \vec{R}_k(\vec{\Theta})\right] \frac{\exp(i\,\frac{2\pi}{\lambda}\,r)}{r} \tag{10}$$

Further, the power flowing towards the radiating structure is the sum of the powers flowing towards it in the individual lines,

$$P_{total} = \sum_k |x_k|^2 = \sum_k x_k^* x_k \tag{11}$$

In matrix notation, we define the transpose of the column matrix x by the row matrix x^T,

$$x^T = (x_1, x_2 \; .... \; x_M) \tag{12}$$

and by the usual laws of matrix multiplication,

$$P_{total} = x^{T*} x \tag{13}$$

Similarly, the total power carried by the reflected waves in the feed-lines is simply,

$$P_{refl} = y^{T*} y \tag{14}$$

Now, we can also define the transpose of S,

$$(S^T)_{kj} = S_{jk} \tag{15}$$

And finally, let us now define the conjugate transpose of S, denoted by the matrix, S^+, as

$$S^+ = (S^T)^* = (S^*)^T$$

$$(S^+)_{kj} = S^*_{jk} \tag{16a}$$

and similarly define the conjugate transpose column vectors

$$x^+ = (x^T)^*$$

$$y^+ = (y^T)^* \tag{16b}$$

We also recall that for two matrices, A and B,

$$(AB)^T = B^T A^T \tag{17}$$

Thus we may rewrite Eqs. (13) and (14) as

$$P_{total} = x^+ x \tag{18}$$

$$P_{refl} = y^+ y \tag{19}$$

and recalling Eq. (3),

$$\begin{aligned} P_{refl} &= (Sx)^+ (Sx) \\ &= x^+ S^+ S x \end{aligned} \tag{20}$$

Finally, the total radiated power, using Eqs. (10), (9) and (8), is

$$\begin{aligned} P_{rad} &= \int r^2 \frac{\eta_o}{2} \vec{E}^*(\vec{\Theta}) \cdot \vec{E}(\vec{\Theta}) \, d\vec{\Theta} \\ &= \sum_{k,j=1}^{M} x_k^* q_k^* x_j q_j \left[\frac{\eta_o}{2} \int \vec{R}_k^*(\vec{\Theta}) \cdot \vec{R}_j(\vec{\Theta}) \, d\vec{\Theta} \right] \\ &= \sum_{k,j=1}^{M} x_k^* q_k^* \beta_{kj} q_j x_j \end{aligned} \tag{21}$$

If we further define an M x M matrix, Γ, whose elements are

$$\Gamma_{kj} = q_k^* \beta_{kj} q_j \tag{22}$$

we can write

$$P_{rad} = x^+ \Gamma x \tag{23}$$

The <u>law of conservation of energy</u> is now very simply

$$P_{total} \geq P_{refl} + P_{rad} \tag{24}$$

with the equality sign holding if and only if we are discussing a lossless structure. For at least part of our discussion, we shall not make any such restriction; our equations above are perfectly valid even if absorption is taking place within the structure, so long as the system remains linear in the general sense (in the sense of superposition holding for all voltages and currents). We may also note that although for any linear passive network, the S matrix will be symmetric ($S_{kj} = S_{jk}$) by the law of reciprocity, our forms above have not included this requirement, and our equations include the possibility of non-reciprocal systems (non-symmetric S), provided, again, they are still linear in the sense of superposition remaining valid. With these comments in mind, we finally rewrite Eq. (24) in terms of Eqs. (18), (19), and (23), as*

$$x^+x \geq x^+S^+Sx + x^+ \Gamma x \tag{25}$$

and this equation must hold for <u>any</u> arbitrary choice of the excitation voltages, x.

---

*I am indebted to my colleague, Dr. J. E. Storer, for pointing out the essential matrix nature of the fundamental equations, following upon my original derivations. Many of the results below would not have been obtained without this recognition, and the resulting compactness of the mathematics. The general fundamental result in Eq. (39) was pointed out by Dr. Storer, immediately upon this recognition. He also pointed out the possibility for extending the studies to discussions of the scattering matrix, thus stimulating and contributing to the further investigation reported in the companion paper, Part II.[1]

We may note that, if we introduce the identity matrix, I,

$$(I)_{kj} = \delta_{kj} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases} \tag{26}$$

we can rewrite Eq. (25) as

$$x^+(I - S^+S - \Gamma)\, x \geq 0 \tag{27}$$

and, since this holds for arbitrary x, it states that $(I - S^+S - \Gamma)$ must be a positive semi-definite matrix for a lossy network, and must be the zero matrix if and only if a lossless structure is involved. As will be shown immediately, it is also Hermitian, and this is sufficient to provide much valuable information. However, we will ultimately wish to return to Eq. (25), and hence will continue to use it directly throughout our derivations.

5. Limitation on Radiation Efficiency Implied by the Conservation of Energy Equations:

Let us first note that both Γ and the matrix

$$W = S^+S \tag{28}$$

are Hermitian, which is to say that

$$\begin{aligned} \Gamma^+ &= \Gamma \qquad & (\Gamma^+)_{kj} &= \Gamma_{kj} \\ W^+ &= W \qquad & (W^+)_{kj} &= W_{kj} \end{aligned} \tag{29}$$

The second of these follows directly from Eq. (28),

$$W^+ = (S^+S)^+ = (S^+)(S^+)^+ = S^+S = W$$

and the first from the definition of Γ in Eq. (22), and the symmetry property of β in Eq. (9),

$$(\Gamma^{+})_{kj} = \Gamma^{*}_{jk} = (q^{*}_{j}\,\beta_{jk}\,q_{k})^{*} = q_{j}\,\beta^{*}_{jk}\,q^{*}_{k} = q^{*}_{k}\,\beta_{kj}\,q_{j} = \Gamma_{kj}$$

Secondly, both Γ and W are positive semi-definite, i.e., for any vector z,

$$\begin{aligned} z^{+}\,\Gamma\,z &\geq 0 \\ z^{+}Wz &\geq 0 \end{aligned} \tag{30}$$

The second of these follows almost immediately again from Eq. (28)

$$z^{+}Wz = z^{+}S^{+}Sz = (Sz)^{+}(Sz) \geq 0$$

(i.e., for any column vector t, $t^{+}t = \sum_{k=1}^{M} |t_k|^2 \geq 0$, with equality only if all $t_k = 0$).
The first follows likewise by noting from Eqs. (22) and (8),

$$z^{+}\,\Gamma\,z = \sum_{k,j=1}^{M} z^{*}_{k}\,q^{*}_{k}\,\beta_{kj}\,q_{j}\,z_{j}$$

$$= \frac{\eta_o}{2} \sum_{k,j=1}^{M} \int z^{*}_{k}\,q^{*}_{k}\,\vec{R}^{*}_{k}(\vec{\Theta}) \cdot \vec{R}_{j}(\vec{\Theta})\,q_{j}\,z_{j}\,d\vec{\Theta}$$

$$= \frac{\eta_o}{2} \int \left[\sum_{k=1}^{M} z_k\,q_k\,\vec{R}_k(\vec{\Theta})\right]^{*} \cdot \left[\sum_{k=1}^{M} z_k\,q_k\,\vec{R}_k(\vec{\Theta})\right] d\vec{\Theta} \geq 0$$

since the product in the integral is now the square of the absolute value of the magnitude of a complex vector. Now, with both W and Γ known to be positive semi-definite and Hermitian, we know from matrix theory the important property that each has exactly M positive real or zero eigenvalues, and M corresponding mutually orthogonal (complex) eigenvectors. We also know that there exists a unitary matrix U, whose columns are a set of orthonormalized eigenvectors of Γ, such that*

$$U^{+}U = I, \text{ or } U^{+} = U^{-1} \quad \text{(unitary property)} \tag{31}$$

and

$$U^{+}\Gamma U = \gamma \tag{32}$$

where γ is a purely diagonal matrix. That is, the elements of γ on the main diagonal are the eigenvalues of Γ, and the off-diagonal elements are 0,

$$\gamma_{kj} = \delta_{kj}\,\gamma_k, \qquad \gamma_k \text{ positive real or zero} \tag{33}$$

---

*It is important to note that each eigenvector of a matrix is determined only to within a scale factor. The normalization to unit length, i.e., the requirement $z^{+}z = 1$, is necessary if the eigenvectors are to be used to construct a unitary matrix with the properties given in Eqs. (31) and (32). However, the components of z are still completely indeterminate to within a common phase factor of the form $\varepsilon^{i\phi}$. Stated another way, suppose we consider any matrix D which is diagonal, with elements,

$$D_{kj} = \delta_{kj}\,\varepsilon^{i\phi_k} = \begin{cases} 0 & k \neq j \\ \varepsilon^{i\phi_k} & k = j \end{cases},$$

i.e., D of the form

$$D = \begin{pmatrix} \varepsilon^{i\phi_1} & & 0 \\ & \varepsilon^{i\phi_2} & \\ 0 & & \varepsilon^{i\phi_M} \end{pmatrix}$$

where $\phi_1, \phi_2, \ldots \phi_M$ are arbitrary real numbers. Then it is readily verified that any such D itself is unitary and that any matrix V of the form

$$V = UD$$

is also unitary, and also diagonalizes Γ into γ. Forms like D are encountered in Part II in attempting to characterize the form of the scattering matrix.

If we return to Eq. (25), and introduce U as just defined by transforming from x to a new (equally arbitrary) vector z,

$$
\begin{aligned}
x &= Uz \\
x^+ &= z^+U^+
\end{aligned}
\tag{34}
$$

we can rewrite Eq. (25) as

$$z^+U^+Uz \geq z^+U^+S^+SUz + z^+U^+ \Gamma Uz$$

Or, using Eqs. (31) and (32)

$$z^+z \geq z^+U^+S^+Uz + z^+\gamma z \tag{35}$$

We may rewrite this inequality as

$$z^+(I-\gamma)z \geq z^+Tz \tag{36}$$

where

$$T = U^+S^+SU \tag{37}$$

is readily observed to also be Hermitian, and positive semi-definite. Now the right-hand side of Eq. (36) is positive or zero, for arbitrary z, while on the left-hand side, the matrix $(I-\gamma)$ has non-zero elements only on the diagonal. It follows from the arbitrariness of z under which Eq. (36) holds, that these diagonal elements of $(I-\gamma)$ must hence individually be positive or zero, i.e., we must be able to write

$$(I-\gamma)_{kj} = \lambda_k^2 \delta_{kj} \tag{38a}$$

where

$$\lambda_k^2 = 1 - \gamma_k \geq 0 \tag{38b}$$

This is an important result. In particular, Eq. (38b) implies that the <u>largest</u> of the eigenvalues of Γ cannot exceed unity,

$$(\gamma_k)_{max} \leq 1 \tag{39}$$

If we recall the form of Γ, i.e.,

$$\Gamma_{kj} = q_k^* \beta_{kj} q_j$$

where the β_{kj} are fixed by the design of the beams, we see that Eq. (39) places <u>direct limitations</u> on the values of the beam radiation efficiencies, $|q_k|^2$. In a companion paper,[1] we derive further from the conservation of energy equations, some of the properties of the scattering matrix, S. In the remainder of this paper we will discuss, via examples, the physical meaningfulness of the result in Eq. (39).

6. <u>Example</u>: <u>Equal Beam Efficiencies (Angle Diversity Receiving Arrays)</u>

An important and simple result is obtained from Eq. (39) for the case when all the beams are designed to be a priori equal, i.e.

$$q_k = q \quad \text{for all } k \tag{40}$$

This has been the usual design case, for example, in angle diversity receiving arrays of the kind under recent study in connection with tropospheric scatter reception;[2]-[6],

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(2) A.B. Crawford, D.C. Hogg, and W.H. Kummer, "Studies in Tropospheric Propagation Beyond the Horizon," BSTJ, 38, 1067-1178 (1959).

(3) J.H. Vogelman, J.L. Ryerson, and M.H. Bickelhaupt, "Tropospheric Scatter System Using Angle Diversity," Proc. IRE, 47, pp. 688-696 (1959).

(4) R. Bolgiano, Jr., N.H. Bryant, and W.E. Gordon, "Diversity Reception in Scatter Communications with Emphasis on Angle Diversity," Cornell Univ., Final Report, Part I, on AF30(602)-1717, Jan., 1958.

(5) C. Chu, A.H. Wren, and J. LaRue, "Evaluation of the Pincushion System," Univ. Mich. Rad. Lab. Rept. 2872-1-T, RADC-TN-60-50, Feb. 1960.

(6) W.R. Richard and M.H. Bickelhaupt, "Multiple Angle Diversity Design Considerations," Rome Air Development Center, RADC-TN-60-22, April, 1960.

and it was, in fact, difficulties which were first noticed during a related study[7] which originally motivated the present investigation. In disucssing a receiving array, rather than a transmitting array, we are invoking the well-known reciprocity relationship between the receiving and transmitting antennas, which can be readily shown to apply to the radiation efficiency defined by $|q_k|^2$ above. I.e., if all M beams are allowed to receive simultaneously with matched receivers, one may calculate the actual power received by each receiver by first calculating the power which would be received according to its corresponding beam pattern, as if no other beams existed, and then multiplying by the radiation efficiency $|q_k|^2$ for that beam.*

Returning to Eq. (40), we see that it implies

$$\Gamma_{kj} = |q|^2 \beta_{kj} \tag{41}$$

It then readily follows that the $\{\gamma_k\}$, the eigenvalues of Γ, are directly related to $\{\beta_k\}$, the set of eigenvalues of β, by

$$\gamma_k = |q|^2 \beta_k \tag{42}$$

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(7) D.E. Johansen and S. Stein, "Theory of Antenna Performance in Scatter-Type Reception," to be published in IRE Trans. PGAP; also see report under same title by Stein, Johansen, and Starr, Hermes Electronics Co., ARCRC-TR-59-191, 30 September 1959.

* A note of warning: In many of our results, there is no limitation on whether or not the antenna system is reciprocal. However, whenever we discuss a receiving array in terms of transmitting properties, the entire antenna system is being assumed to be reciprocal. In an actual case of a non-reciprocal antenna system, one will generally want to consider the receiving properties in terms of an equivalent reciprocal transmitting antenna; where, however, it may then no longer be possible to relate receiving properties to measurements made with the real antenna system transmitting.

and Eq. (39) becomes

$$\left|q\right|^2 \leq \frac{1}{(\beta_k)_{max}} \tag{43}$$

Thus, in this case, the inherent limitations due to conservation of energy are completely related to the array of beam cross-coupling factors. Incidentally, it is well known that for any Hermitian matrix, the sum of the eigenvalues equals the sum of the diagonal elements (the trace of the matrix). Thus the largest eigenvalue exceeds the average of the diagonal elements, so that since all diagonal elements of β have the same value $\beta_{kk} = 1$, $\left|q\right|^2 < 1$ is implied; and $\left|q\right|^2 = 1$ is possible only if all eigenvalues are equal, which in turn is readily observed to require that all non-diagonal elements of β vanish, i.e., no beams overlap (e.g., see the next section).

In Appendix 1, we discuss some specific numerical examples which indicate the order of magnitude of the loss in radiation efficiency, for what we regard as some very realistic antenna beam patterns for angle diversity. In particular, we note that although previous angle diversity analyses have considered the effects of beam overlap with respect to correlation in observed propagation-induced fluctuations at the beam output receivers (see also Section 8), they have generally ignored the (basically independent) question of the unavoidable "insertion losses" in the antenna junction whose existence has been shown above. A general statement justifying such an approach seems to be the assertion that if beams are squinted so as to cross over at the 3 db points or lower, then in effect they represent essentially independent receptors of power. The inconsistency of this assumption is, we feel, evident from Appendix 1; the numerical results, nevertheless, we found to be somewhat startling.

In addition, we refer the reader interested in angle diversity to the further discussion in Section 8, and the interesting numerical example and discussion in Appendix 3.

7. Example: Two Limiting Cases

As further relevant examples, we consider two special cases:

(a) No beam overlap.

In this case

$$\begin{aligned} \beta_{kk} &= 1 \\ \beta_{jk} &= 0 \quad , \qquad j \neq k \end{aligned} \tag{44a}$$

Thus, for this case

$$\Gamma_{kj} = \begin{cases} |q_k|^2 & k = j \\ 0 & k \neq j \end{cases} \tag{44b}$$

and, by inspection, the eigenvalues of Γ are exactly

$$\gamma_k = |q_k|^2 \tag{44c}$$

Thus if all $|q_k|^2$ are equal, all beams can be made to have 100% efficiency ($|q|^2 = 1$), as one might expect physically.

A further interesting point now arises from this example. Suppose even though the beams are uncoupled, the design aims at different $|q_k|^2$ for each beam. Then one or more beams corresponding to the largest value will have 100% efficiency, but the others will not. That is, although potentially all beams could be made 100% efficient in this non-overlap situation, a poor a priori choice of the relative q_k would imply building into the system cross-couplings which are not "necessary". This may appear to be a very peculiar kind of statement; nevertheless, it appears that there are many more general antenna system engineering problems where such a priori choices must be made (usually the choice being that all q_k are equal, perhaps without realization that alternatives exist). A further discussion of this issue is given in the next section.

In addition, in Appendix 2, we indicate how one might extend these non-overlap results, via perturbation theory, to a case of much practical design interest, namely that in which the amount of beam coupling between all beams is small.

(b) Complete Beam Overlap

From a mathematical point of view, it is illuminating to consider the other extreme limiting case where all beams overlap completely. Physically, this corresponds to having several "taps" at some point down the line from a single feed. That is, we really do not have separate beams, and certainly cannot extract at each tap an amount of power equal to the total power received by the beam; yet we insist that a correct mathematical formulation, such as we claim to now have, must be capable of fully and correctly accounting for this situation (this simple limiting example, again, being one of the source points for the entire investigation).

For complete overlap, we have

$$\left.\begin{aligned} \beta_{jk} &= 1 \\ \Gamma_{jk} &= q_k^* \, q_j \end{aligned}\right\} \text{ for all } j, k \tag{45}$$

It is readily determined that the eigenvalues of Γ comprise an (M-1)-fold degenerate zero eigenvalue, and one with the value

$$\gamma = \sum_j \left| q_j \right|^2 \tag{46}$$

Thus, without discussing the individual beam radiation efficiencies, we see immediately from Eq. (39) that we have

$$\sum_j \left| q_j \right|^2 \leq 1 \tag{47}$$

which is just what our physical intuition requires. While there is nothing really startling in our achieving this result, since we assume our mathematics have been correct, it is quite satisfying to know that the formulation routinely describes this limiting case, as well as other cases not quite so simple. We are not aware of any previous mathematical description for this general problem which has this satisfying property between all extremes.

8. Example: Specification of Relative Radiation Efficiencies (Application to Optimum Angle Diversity)

Let us return to the general result of Eq. (39), and consider, for a general form of the matrix, β, the question of specifying relative radiation efficiencies. To emphasize what is being specified, it is illuminating to introduce these relative values as dimensionless ratios, K_i, and to assume that all the q_i are now in the form

$$q_i = K_i q \tag{45a}$$

The elements of Γ are thus in the form

$$\Gamma_{ij} = |q|^2 K_i^* \beta_{ij} K_j \tag{45b}$$

$$= |q|^2 \mathcal{K}_{ij} \tag{45c}$$

Thus, if we can find the eigenvalues, $\mathcal{K}_k$, of the matrix $\mathcal{K}$, we will have

$$\gamma_k = |q|^2 \mathcal{K}_k \tag{46a}$$

and from Eq. (39)

$$|q|^2 \leq \frac{1}{(\mathcal{K}_k)_{max}} \tag{46b}$$

At this time, this much is straightforward; however, we regard the availability of an arbitrary pre-specification of the K_i as constituting an extremely interesting problem area. It appears, moreover, that it is precisely with this problem that one necessarily makes contact with the question of the use of the multiple-beam antenna system. In many applications, such as angle diversity referred to above, or the case of a phased array in which only one beam is active at a time, the question of the radiation efficiency of a single beam and of individual line cross-couplings is paramount. In other applications, more than one transmitting beam may be excited at a time, and it is then the performance of the superposition of coupled currents which is relevant. Further, for example, in examining angle diversity for tropospheric scatter systems, one would want to choose the K_i to maximize the diversity performance; in other systems, one might want to impose a constraint of maximum (or minimum) antenna system directionality in certain directions; in others, an approximation within specified tolerances to some composite beam pattern, with maximum power transmitted or received, might be the goal.

That this problem of the optimum choice of the K_i is important we have already seen from our discussion of the zero coupling case in the previous section. It was pointed out that, if all beams are decoupled, any choice of unequal K_i would lead to less than 100% radiation efficiency for some of the beams, but that in this fully decoupled case there was no inherent reason for making such a poor choice. While for this particular problem the correct choice was obvious on physical grounds (see also the discussion of the weak coupling case in Appendix 2), it is not at all apparent as to what might be lost in a more general situation where an arbitrary choice (say all equal) is made for the K_i. Thus the general unsolved problem is to learn more about the implications of the relative values of the q_i, as they affect the γ_i, and in turn through the latter (the conservation of energy constraints), the absolute scale on all the q_i.

As an interesting example of what such a study might lead to in a particular configuration, we can formulate the problem in the angle diversity case. (Unfortunately, we cannot solve the problem in general. Nevertheless, as we indicate later by an example in Appendix 3, in many particular cases, solution may be possible without exorbitant effort.) The quantity of direct interest in these problems is either the probability distribution of the fluctuations in received power or some closely related system performance statistic. The basis of angle diversity as applied to the fading signals of tropospheric scatter is the desire to take advantage of experimental observations that the instantaneous angle of arrival of the waves appears to wander. A number of receiving beams are thus used, each pointing in a different direction, and their outputs combined via an appropriate network. The best combining rule, if thermal noise fluctuations are independent from receiver to receiver, is so-called maximal-ratio combining, the effect of which is, assuming equal noise in all receivers, to add the effective powers of all the receivers. The output noise level, on the assumption made, is the same as that in any single receiving channel. Further, assuming that the fluctuations in signal are described by so-called Rayleigh envelope fading, possibly correlated between receivers due to beam overlap, the characteristic function (Laplace transform of the probability density) has the form[8]

$$F(z) = \frac{1}{\det(I + zL)} \tag{47}$$

Here z is the transform variable, while L is the M x M complex covariance matrix among the complex characterization (a band-pass system is assumed) of the fluctuation

[8] J.N. Pierce and S. Stein, "Multiple Diversity with Nonindependent Fading," Proc. IRE, 48, pp. 89-104 (1960).

process at each receiver. In the parlance of our present investigation,

$$L_{kj} = q_k^* q_j \, \overline{e_k^* e_j} \tag{48}$$

where the bar indicates a statistical average over the fading fluctuations, and each e_j is the (complex) voltage (effective rms into a 1 ohm load) which would be obtained at the receiver output if the j^{th} beam were received with its specified pattern, with no other beams present in the configuration (no cross-coupling to other beams). It has further been pointed out(7) that for non-coherent FSK digital data communications, $\frac{1}{2} F(z)$ in Eq. 47 gives the average bit error rate, if z is taken to be $z = \frac{1}{2N_o}$, where N_o is the noise level. Better yet, if all signal levels are normalized to, say, a median level, z can be taken as proportional to the median-signal noise ratio (see Appendix 3). Hence F(z) in Eq. (47) may in itself, for a specific z, be considered a useful performance measure for the system. In writing Eqs. (47) and (48), there is no presumption that the average level of the fading signals in the respective receivers (the $\overline{e_k^2}$) are the same; in general, depending upon the nature of the transmitting antenna pattern and the actual receiving array configuration, these average levels will differ. Finally, under very reasonable assumptions(7), it is permissible to assume that one can define an apparent distribution of fluctuating sources "in the sky" equivalent, in reception, to the actual fields set up by the transmitting antenna. In terms of the intensity distribution (mean square value at each direction) of this equivalent source distribution, one can write

$$\overline{e_k^* e_j} = \frac{\eta_o}{2} \int \vec{R}_k^*(\vec{\Theta}) \cdot \vec{R}_j(\vec{\Theta}) \, g(\vec{\Theta}) \, d\vec{\Theta} \tag{49}$$

where $g(\Theta)$ is the aforementioned intensity distribution and the $\left\{\vec{R}_k(\vec{\Theta})\right\}$ are the normalized beam patterns defined earlier by Eqs. (4) and (5). If we thus define an array of "beam correlation factors" by the equation

$$G_{kj} = \frac{\eta_o}{2} \int \vec{R}_k^*(\vec{\Theta}) \cdot \vec{R}_j(\vec{\Theta})\, g(\vec{\Theta})\, d\vec{\Theta} \quad , \tag{50}$$

we can write the matrix L in the form

$$L_{kj} = q_k^* \, G_{kj} \, q_j \tag{51}$$

(quite analogous to the form of the Γ -matrix). We can further introduce again the dimensionless K_i (Eq. 45a), and write

$$L_{kj} = |q|^2 \, K_k^* \, G_{kj} \, K_j \tag{52}$$

Thus the over-all optimization of the angle-diversity system, assuming the transmitted field and the receiving beams to be already specified and fixed, consists in finding that set $\{K_j\}$ which, along with the associated best possible value for $|q|^2$ (Eqs. 45-46), minimizes F(z) in Eq. (47) for some specified z. The implication of this statement of the problem is that with a specified transmitted field and array of receiving beams, it might be optimum to enhance the efficiencies of some of the beams at the cost of degrading others, to achieve an over-all optimum performance. The formulation above seems to provide a precise mathematical statement of a problem which has been remarked upon only superficially, at best, in previous studies. The solution to this problem, with further optimization of the receiving beam array configuration and nature of the transmitted field, seems likely to provide useful information as to the true effectiveness of angle diversity systems.

In Appendix 3, we consider a typical angle diversity receiving array by way of illustration, with some interesting conclusions (at least for the numbers chosen).

9. Conclusions and Unsolved Problems

A general formulation has been given which points up the importance and some of the implications of conservation-of-energy relationships in multiple-beam antenna systems. These implications appear as limitations on radiation efficiencies for individual beams. In Part II,[1] further implications are given in terms of characterizing the form of the scattering matrix, i.e., the array of feed-line self- and cross-coupling factors. These results, together, appear quite important for elucidating inherent antenna system design limitations following upon particular choices of beam patterns in such multiple-beam systems.

It appears that further useful results remain to be obtained. These unsolved problems appear to lie in better understanding of certain matrix problems, and in applying maximum ingenuity towards analyzing particular classes of matrices. With respect to the material discussed here in Part I, the major aspect which we regard as unsolved is the significance of the *a priori* choices of the relative beam radiation efficiencies (e.g., the K_i defined in Section 8). In particular, the open question is whether one can determine optimum choices, using relevant criteria as to the desired system performance. Even in the angle diversity case cited, where the problem has been reduced to a specific mathematical statement, one does not have a general solution. For other systems, the deduction of the mathematical statement from which optimization can be determined may represent, in each case, a formidable problem in itself.

Acknowledgements:

As already indicated in a footnote, this investigation has profited greatly from valuable mathematical insights provided by Dr. James E. Storer. In addition, this study has been part of a larger collaborative study, to be reported elsewhere, in which Mr. Donald E. Johansen has been a principal investigator; our many discussions contributed significantly towards grasping the nature and significance of the underlying physical principles explored in this paper.

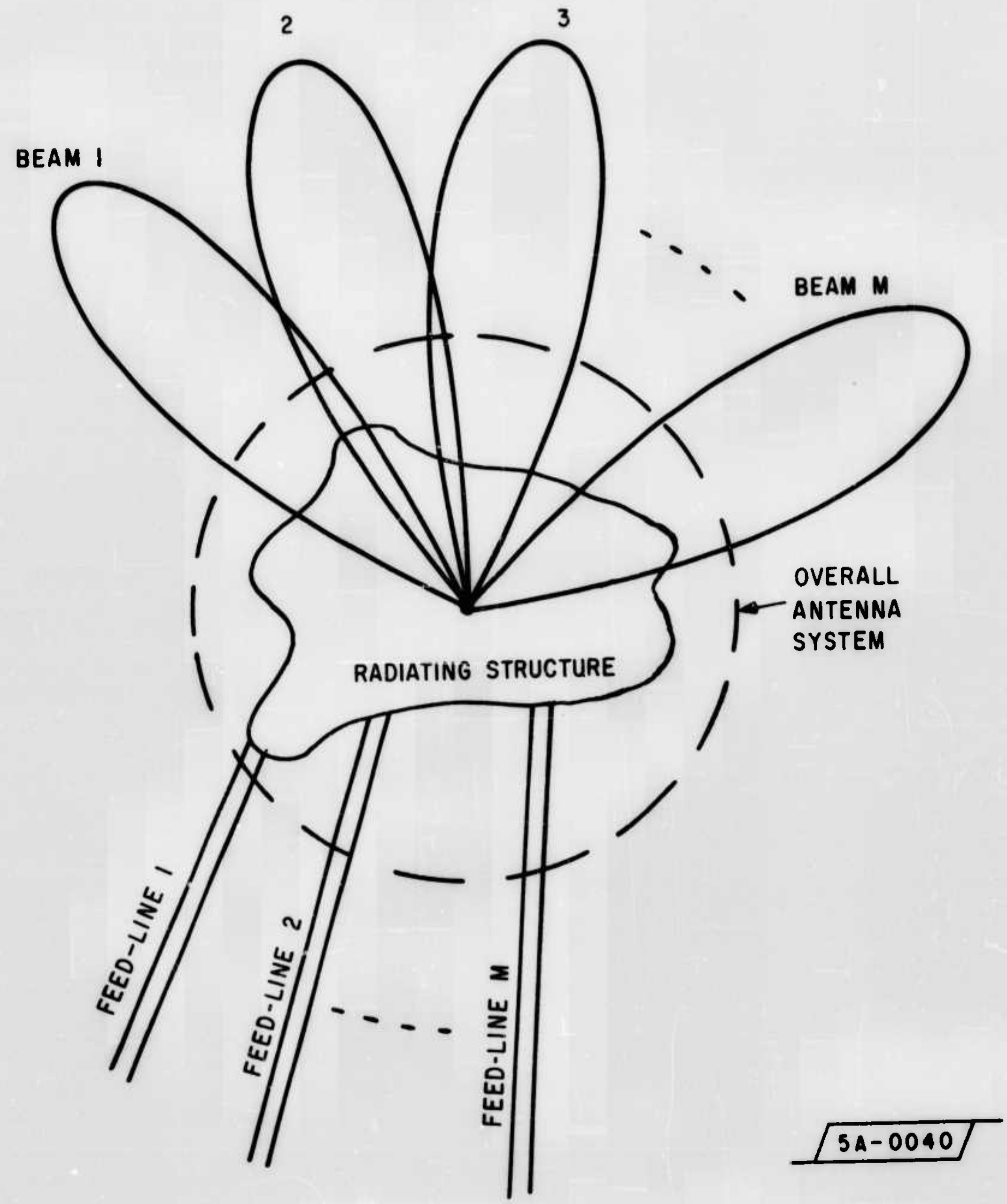

Figure 1. Multiple Beam Antenna System

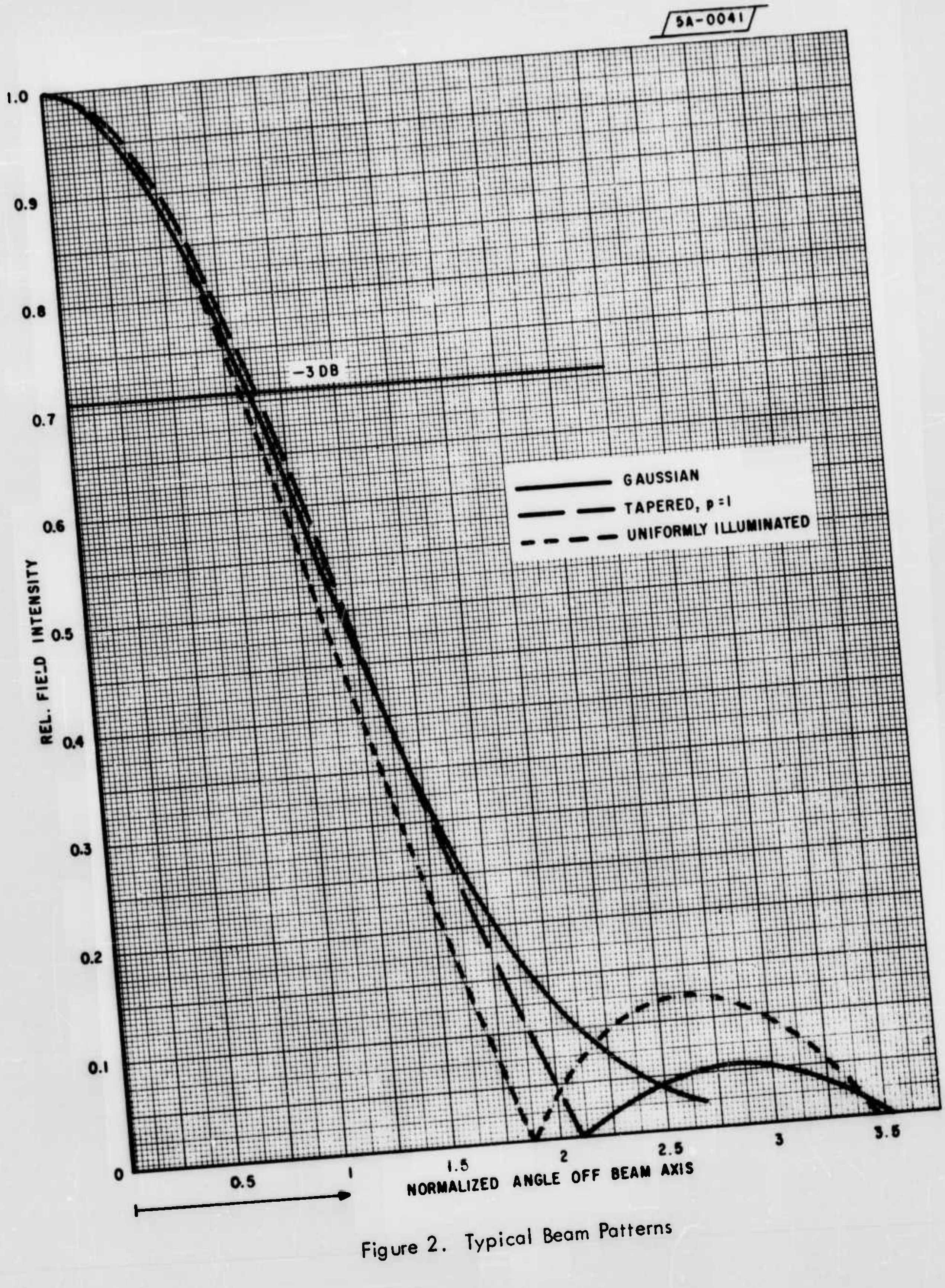

Figure 2. Typical Beam Patterns

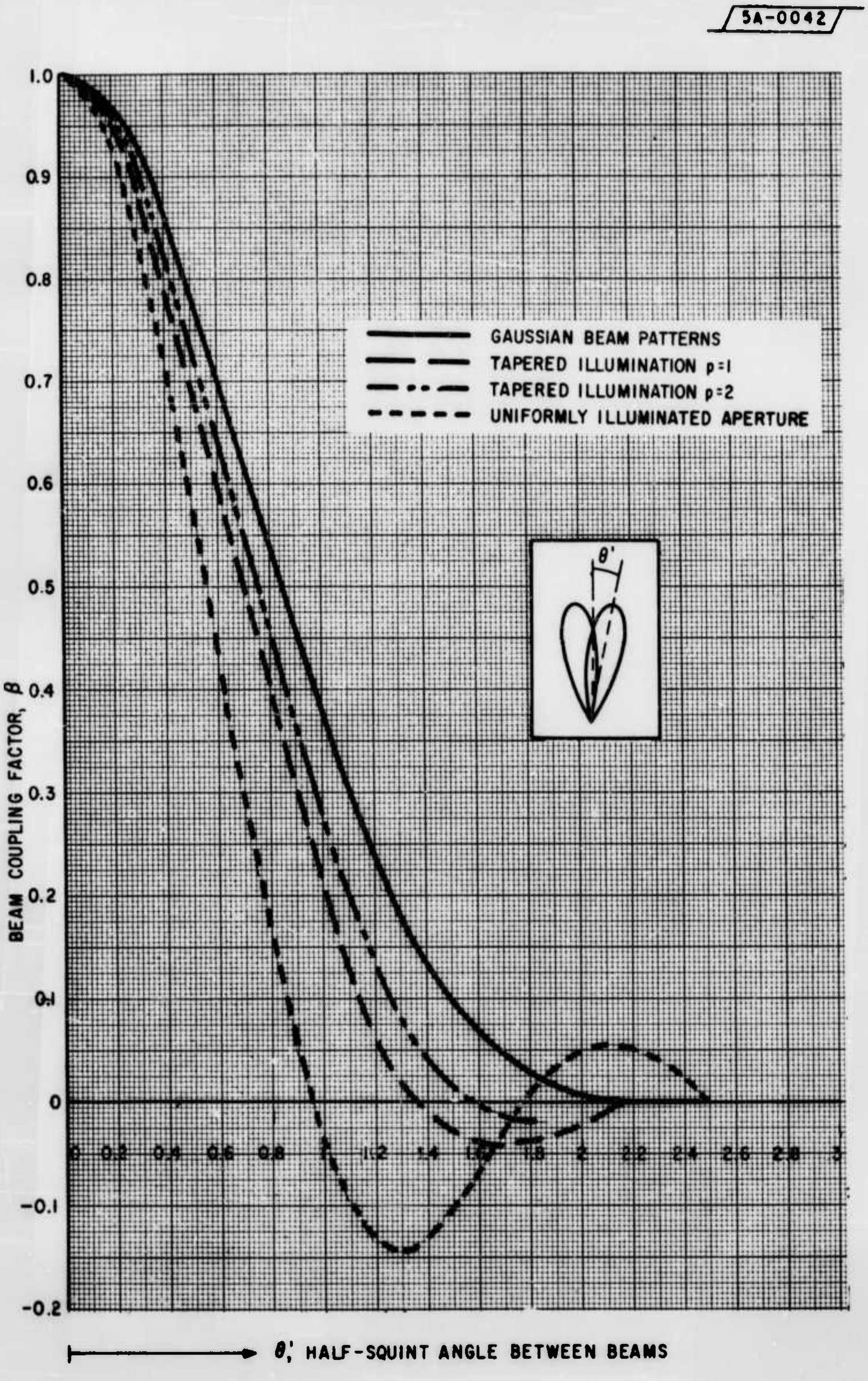

Figure 3. Beam-Coupling Factors

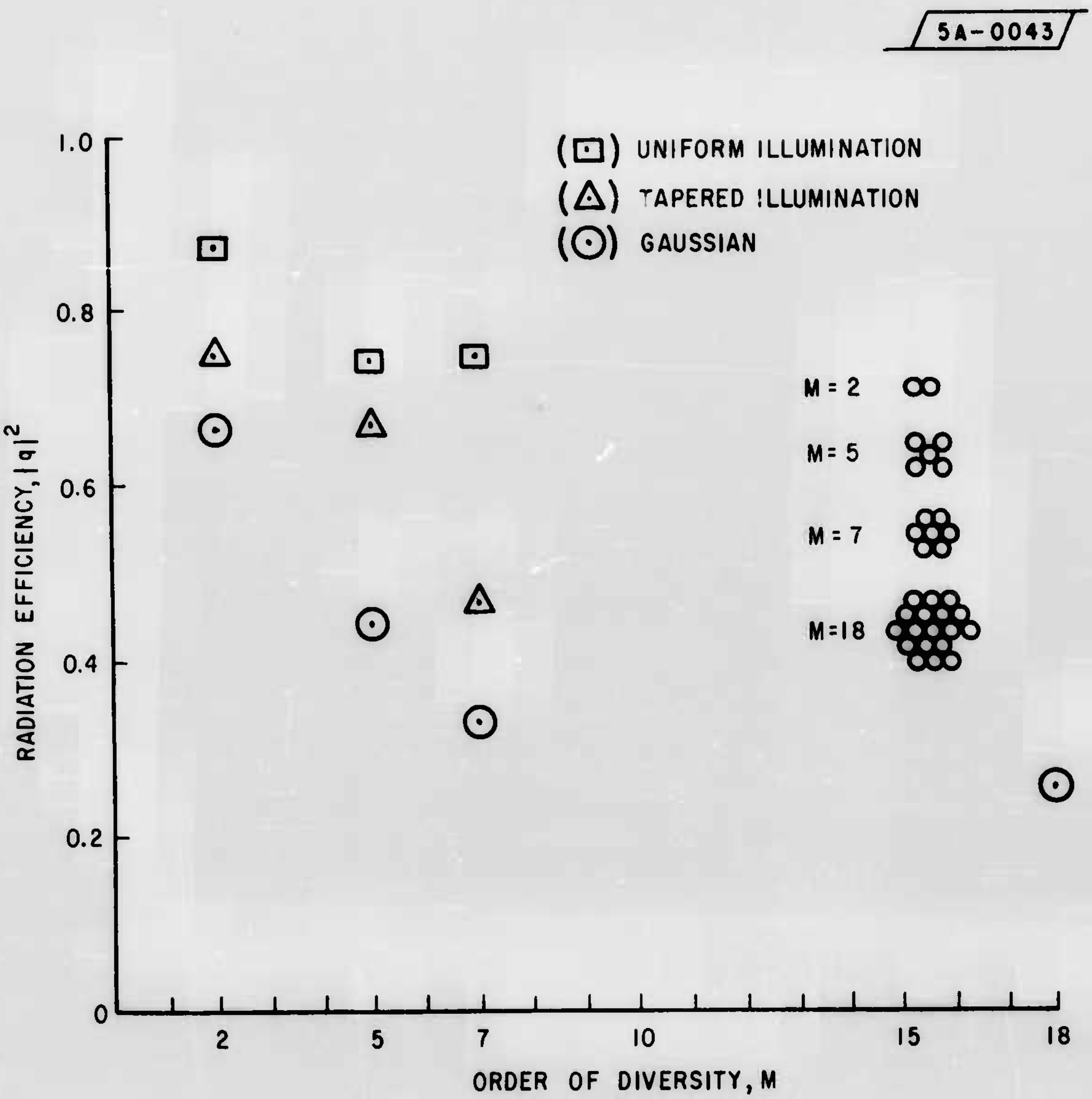

Figure 4. Radiation Efficiency for Angle Diversity Beams

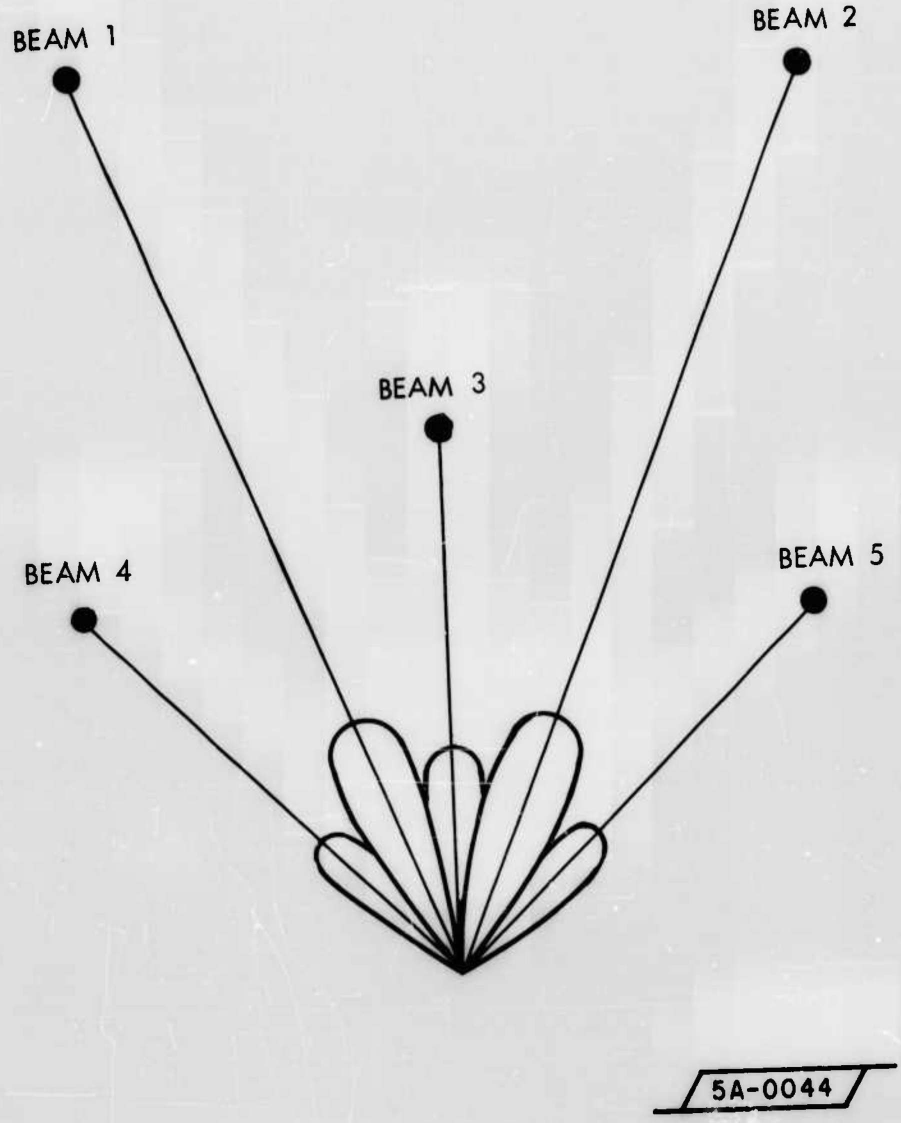

Figure 5. Typical Angle Diversity Configuration

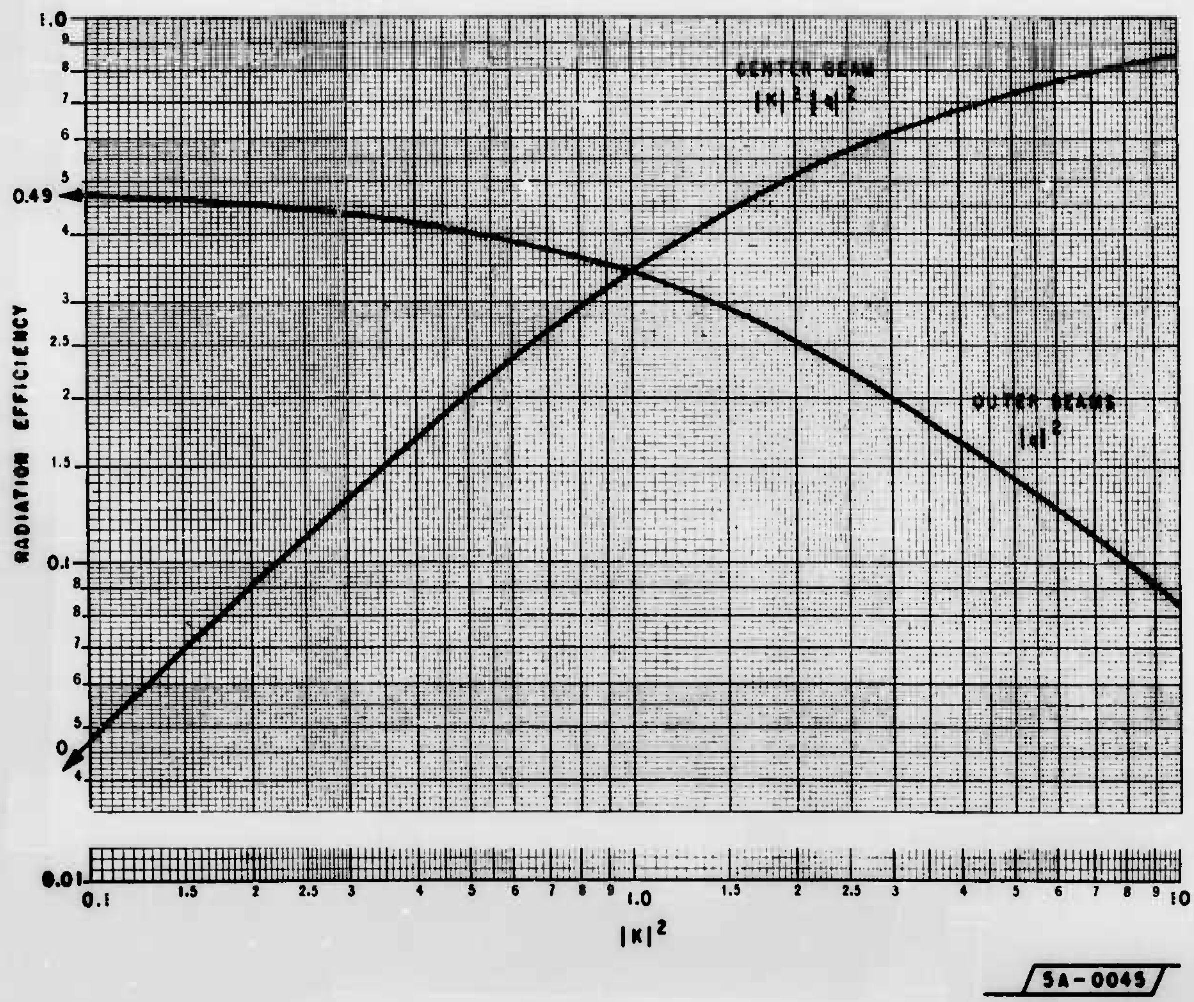

Figure 6. Radiation Efficiency Versus $|K|^2$

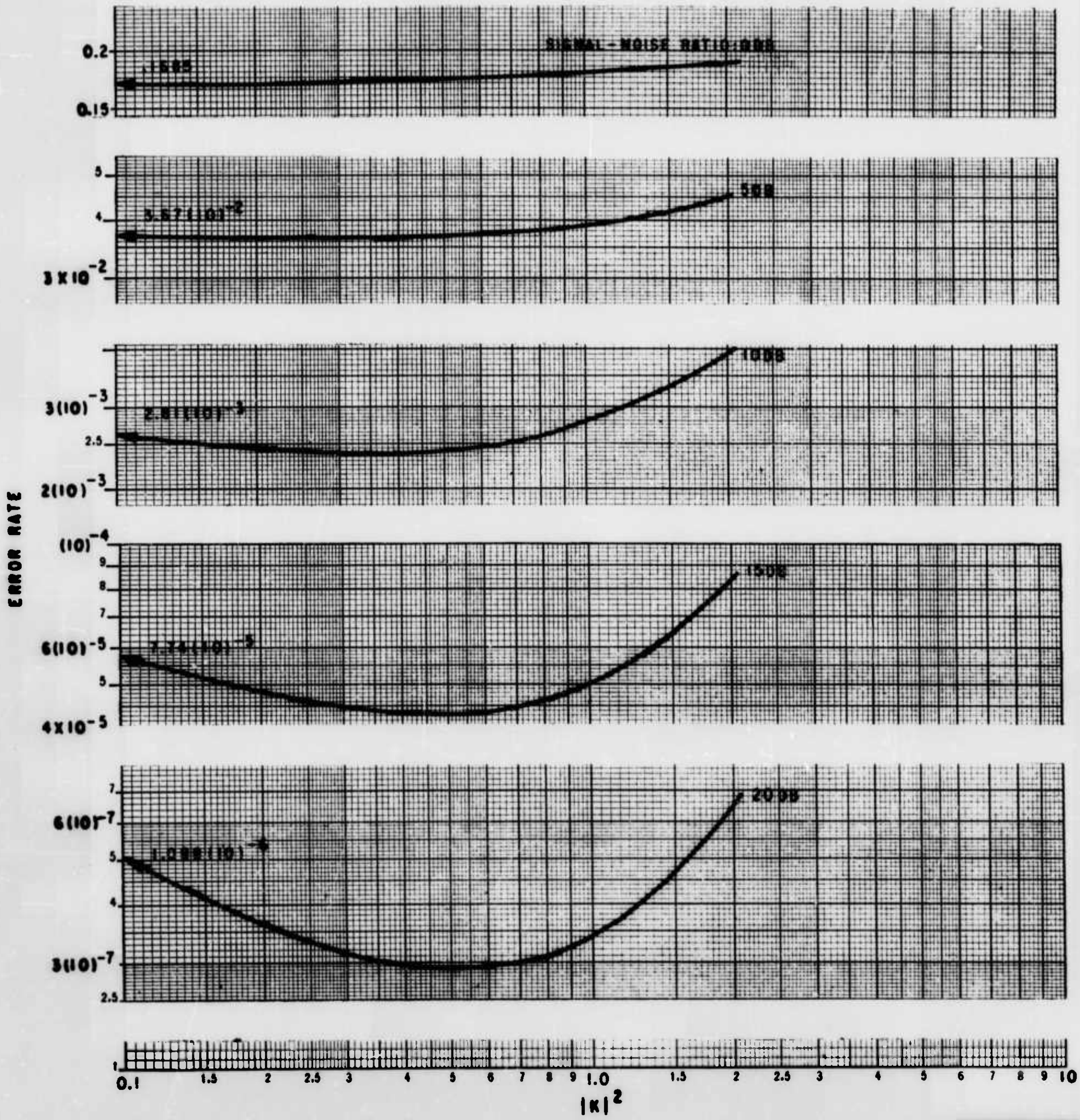

Figure 7. Error Rate Versus $|K|^2$; $\theta_o = 1$

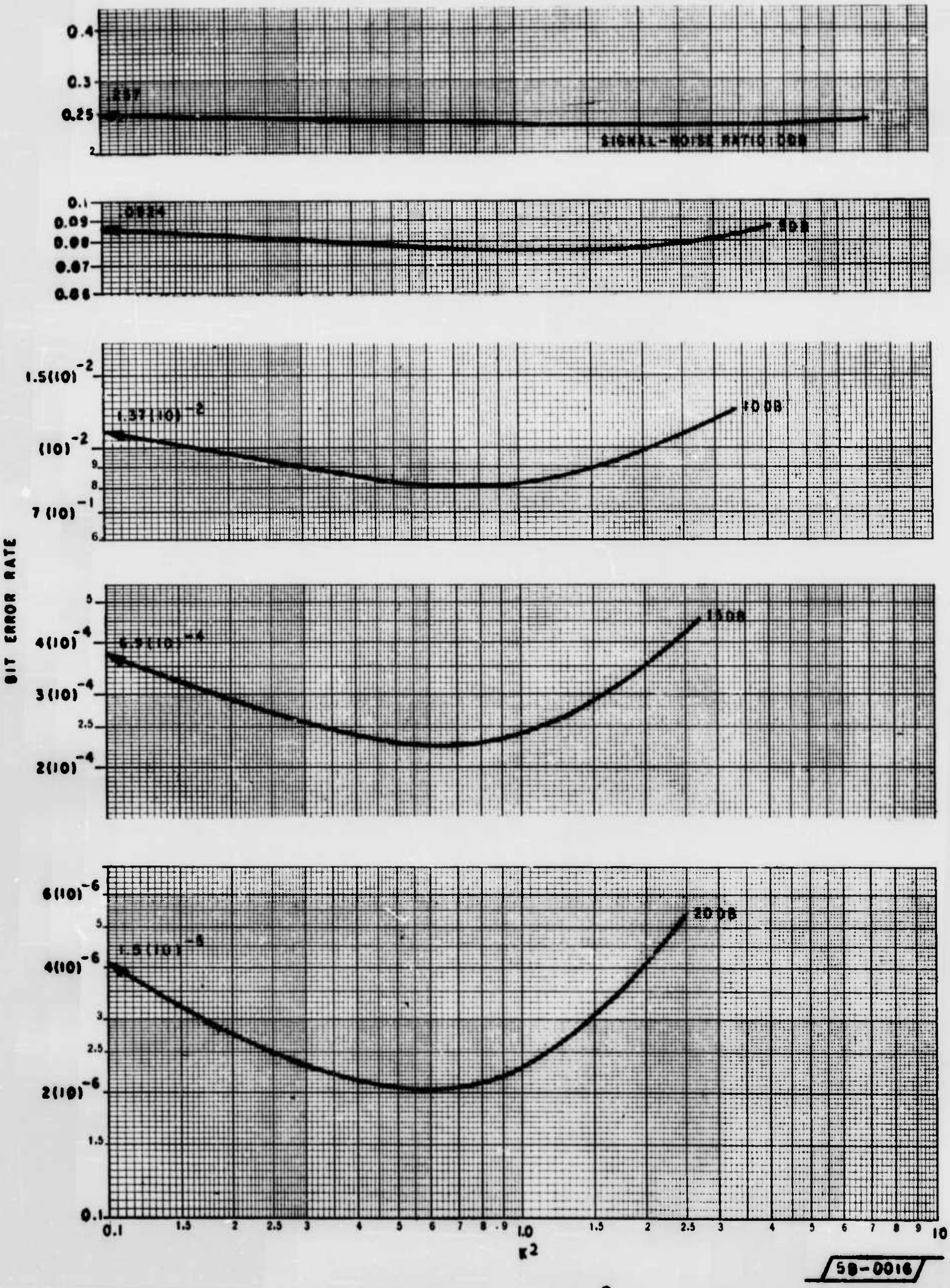

Figure 8. Error Rate Versus $|K|^2$; $\theta_o = 2$

APPENDIX 1: A NUMERICAL EXAMPLE FOR RADIATION EFFICIENCY*

In Figure 2, we show three beam patterns of interest (field patterns) for a circular aperture with linear polarization. Two of these are idealized patterns: These are for the "uniformly illuminated" aperture of radius a, and the gaussian-illuminated aperture (with the aperture currents extending to infinity). The third is a tapered-aperture distribution of the form $\left(1 - \frac{r^2}{a^2}\right)^p$, with p = 1. For all three, the parameters of the antenna were chosen so that the main lobes are essentially of identical width at the 3 db points. The tapered distribution represents a good approximation to actual designs which aim at lower side-lobe levels than associated with the uniformly-illuminated aperture, while at the same time avoiding the losses due to spillover (which do not appear in these theoretical calculations).

We assume next that for each type of beam, a multiple-beam system is constructed with a common aperture, with the various "squint" angles (directions of pointing) obtained by appropriate uniform phasing of the currents across the apertures. With this assumption, we can readily compute the beam-coupling factors between any pair of beams. (Incidentally, the calculations are most readily accomplished by recalling the Fourier Transform relationship between beam patterns and corresponding aperture distributions; and then transforming Eq. (8) as a special case of a convolution integral to give an equivalent integral over the aperture plane.) In Figure 3, we have plotted as a function of the squint angle between any pair of beams of like type, the associated beam-coupling factor, β. An additional

*I am indebted to my colleague, Mr. W. J. Cowan, for providing me with the material of this section, especially Figures 2, 3, and 4; and as well, the numerical data used in the example of Appendix 3.

curve has been added here for the $p = 2$ taper, whose beam pattern would be even closer to the gaussian in Figure 2 than the $p = 1$ pattern. The abscissae in Figure 3 are half the total squint angle between beams, with the same angular scale as in Figure 2. Thus, for beams crossing over at the 3 db points, we see by comparing Figures 2 and 3 that the overlap factor varies from 0.45 for gaussian shaped beams, to about 0.35 for the tapered illuminations, to about 0.1 for the uniformly illuminated case.

For two beams, with equal <u>a priori</u> radiation efficiencies, the eigenvalues of interest are those of the β-matrix, obtained by solving

$$\begin{vmatrix} 1 - x & \beta \\ \beta & 1 - x \end{vmatrix} = 0$$

Or

$$\begin{aligned} x_1 &= 1 - \beta \\ x_2 &= 1 + \beta \end{aligned} \qquad \text{(A1-1)}$$

From Eq. (43), the associated upper-bound radiation efficiency is thus given as

$$\left|q\right|^2_{max} = \frac{1}{1 + \beta} \qquad \text{(A1-2)}$$

$$\begin{aligned} \text{For } \beta &= 0.49, & \left|q\right|^2_{max} &= 0.67 \\ \beta &= 0.3\ , & \left|q\right|^2_{max} &= 0.76 \\ \beta &= 0.14, & \left|q\right|^2_{max} &= 0.88 \end{aligned} \qquad \text{(A1-3)}$$

These are shown as the case $M = 2$ in Figure 4. It is interesting to note that with only the two beams, the efficiencies are already reduced by 25%, or about 1.4 db! In Figure 4, we have also indicated additional radiation efficiencies for particular higher-order configurations for which calculations have been made for other purposes. In all cases, equal a priori radiation efficiencies are assumed (also see, however, Appendix 3). The circles in the sketches of the configurations show how the -3 db contours intersect for the assumed configurations. It is interesting to note that the data do not seem to lie on smooth curves. This is particularly true for the M = 5 case. We attribute this, comparing the sketches of the configurations, to the fact that it is the actual couplings of each beam to adjacent beams which is important, and these are especially different for the M = 5 configuration than for either the M = 7 or M = 18 cases.

Incidentally, the results in Figure 3 indicate quite different beam overlap factors, and hence quite different cross-couplings, for beams which differ only in their side lobe structure but have almost identical main lobes. In one sense, this points out the importance of the side-lobe structure in multiple-beam systems. In another sense, it verifies a common engineering fact, that one may make minor changes in a feed structure which essentially leave the main beam unchanged (presumably, however, radically altering the side-lobes), while radically reducing cross-coupling effects. It is also interesting to note from Figure 3 that those beam patterns (tapered illuminations) which more closely approach the ideal of a monotonic decrease of field off the direction of pointing, are also those which are worst in terms of producing beam overlap and hence cross-coupling effects. In any case, beam overlap effects appear to be far from insignificant.

APPENDIX 2: WEAK BEAM OVERLAP - A PERTURBATION CALCULATION

In this case, we assume for the overlap factors,

$$\beta_{kk} = 1$$
$$\left|\beta_{kj}\right| \ll 1 , \quad j \neq k \tag{A2-1}$$

An array of overlap factors of this type is extremely important from an engineering viewpoint, since it is likely to represent the situation where some overlap is occurring, but every attempt has been made in design planning and because of operational requirements to keep it small. For particular cases, where conditions of symmetry and the like sufficiently simplify the problem, an exact solution may be possible. This is shown, for example, by the illustrative problem treated in Appendix 3. In this section, however, we wish to indicate a more generally applicable method for obtaining an approximate solution in a general case, by applying standard perturbation techniques,(9),(10) with the zero-coupling case in Eq. (44) regarded as the unperturbed case. I.e., we write the Γ-matrix for the weak-coupling situation in the form

$$\Gamma = \Gamma^{(0)} + \epsilon \, \Gamma' \tag{A2-2}$$

where $\Gamma^{(0)}$ is the matrix for the zero-coupling case, Γ' is a matrix whose elements are of the same order as those of $\Gamma^{(0)}$, and ϵ is a perturbation parameter for which it is assumed that

$$\epsilon \ll 1 \tag{A2-3}$$

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(9) L.I. Schiff, Quantum Mechanics, McGraw-Hill, New York, 1949, p. 149-156.

(10) R. Bellman, Introduction to Matrix Analysis, McGraw-Hill, New York, 1960, p. 61-63.

Thus, referring to Eq. (44), we have

$$\Gamma_{jk}^{(0)} = \delta_{jk} \left|q_k\right|^2 \tag{A2-4}$$

Further, for $\Gamma^{(0)}$, we can formalize the results in Eq. (44), by describing its m^{th} eigenvector, u_m, and corresponding m^{th} eigenvalue, γ_m, as satisfying the matrix equation

$$\Gamma^{(0)} u_m = \gamma_m u_m \tag{A2-5a}$$

or, in component form,

$$\sum_k \Gamma_{jk}^{(0)} u_{mk} = \gamma_m u_{mj} \tag{A2-5b}$$

where u_{mk} is the k^{th} component of the m^{th} eigenvector. Referring to Eq. (44), we readily observe that

$$\gamma_m = \left|q_m\right|^2 \tag{A2-6a}$$

and that the detailed components of the corresponding normalized m^{th} eigenvector have the very simple form

$$\begin{cases} u_{mk} = 0 \ , & k \neq m \\ u_{mm} = 1 \end{cases} \tag{A2-6b}$$

In addition to all eigenvectors being normalized to length unity, we may recall that for a Hermitian matrix, they also form a mutually orthogonal set. For the case of a degenerate eigenvalue, there is some arbitrariness in the construction of this mutually orthogonal set; we will discuss this point further below in connection with the perturbation calculation (we have seen already that we may indeed have much physical interest in a case where some or all the q_k are equal in magnitude, in which event the M x M matrix $\Gamma^{(0)}$ has up to an M-fold degeneracy).

The perturbation calculation, in effect, seeks to observe the (presumed) gradual and continuous changes in the eigenvalues and eigenvectors of Γ as the perturbation parameter, ϵ, in Eq. (A2-2) increases slowly from zero. Since the diagonal elements of any Γ-matrix are always just

$$\Gamma_{kk} = \left|q_k^2\right| = \Gamma_{kk}^{(0)} \qquad \text{(A2-7)}$$

we see that any perturbing matrix, Γ', in Eq. (A2-2) must have identically zero elements along the main diagonal,

$$\Gamma'_{kk} = 0 \qquad \text{for all } k \qquad \text{(A2-8a)}$$

Furthermore, although ϵ is to be a parameter in the perturbation calculation, we will always wish to choose as our point of calculation the combination of ϵ and Γ' such that for the off-diagonal elements (recalling that $\Gamma_{kj}^{(0)} = 0,\ j \neq k$),

$$\epsilon\ \Gamma'_{kj} = q_k^* \beta_{kj} q_j \qquad \text{(A2-8b)}$$

where the β_{kj} is that described in Eq. (A2-1).

Now the essence of the perturbation calculation is to assume that the eigenvalues and eigenvectors for the perturbed matrix, Γ, in Eq. (A2-2), differ only slightly from those of $\Gamma^{(0)}$; and that they can be represented as power series expansions in the small parameter, ϵ, with zeroth-order values given by the corresponding forms relative to $\Gamma^{(0)}$. I.e., if the m^{th} eigenvector and eigenvalue of Γ

are z_m (k^{th} component, z_{mk}) and ψ_m respectively, we assume that we can write

$$z_m = z_m^{(0)} + \epsilon z_m^{(1)} + \epsilon^2 z_m^{(2)} + \ldots = \sum_{n=0}^{\infty} \epsilon^n z_m^{(n)}$$

(A2-9)

$$\psi_m = \psi_m^{(0)} + \epsilon\psi_m^{(1)} + \epsilon^2\psi_m^{(2)} + \ldots = \sum_{n=0}^{\infty} \epsilon^n \psi_m^{(n)}$$

(For the usual non-degenerate case, we would take $z_m^{(0)} = u_m$ and $\psi_m^{(0)} = \gamma_m$; we amplify on this below.) If we insert Eq. (A2-9) into Eq. (A2-2), recognize a zeroth-order identity on the basis of Eq. (A2-5a), and collect all terms in like powers of ϵ, we have a term-by-term power series identity relating the higher-order perturbation correction factors. Confining ourselves to the first two orders, we have

First order: $$\Gamma^{(0)} z_m^{(1)} + \Gamma' z_m^{(0)} = \psi_m^{(0)} z_m^{(1)} + \psi_m^{(1)} z_m^{(0)} \qquad \text{(A2-10)}$$

Second order: $$\Gamma^{(0)} z_m^{(2)} + \Gamma' z_m^{(1)} = \psi_m^{(0)} z_m^{(2)} + \psi_m^{(1)} z_m^{(1)} + \psi_m^{(2)} z_m^{(0)} \qquad \text{(A2-11)}$$

At this point, the remainder of the effort consists in recognizing and using the fact that $z_m^{(1)}$ and $z_m^{(2)}$ are themselves vectors in the M-dimensional space spanned by the mutually orthogonal, normalized set $\{u_k\}$, and hence that they can be expanded as a sum over the $\{u_k\}$ with appropriate coefficients. However, first we must consider again the degeneracy problem. It is likely, and indeed common (e.g., the "line-splitting" in quantum-mechanical problems) that under perturbation a zeroth-order degenerate eigenvalue will give rise to non-degenerate (unequal) perturbed eigenvalues. Now, although for a Q-fold degeneracy in an eigenvalue of a Hermitian matrix, Q mutually orthogonal eigenvectors can be constructed, <u>any</u> linear combination of these also satisfies the eigenvalue equation for the particular degenerate eigenvalue, and may substitute as an eigenvector. Thus, although we know that the perturbed eigenvector has as its zeroth-order form <u>some</u> particular such linear combination, we do not know <u>a priori</u> what this form is; generally it will not correspond to a particular <u>one</u> of the set of mutually orthogonal eigenvectors which we have somewhat arbitrarily constructed in the degenerate zeroth-order situation. Thus, for example, if (say) the first Q($m = 1, \ldots Q$) eigenvectors of $\Gamma^{(0)}$ describe such a degenerate eigenstate,* we must replace $z_m^{(0)}$ in Eqs. (A2-9) to (A2-11), for any $m \leq Q$ (i.e., in seeking a perturbed state corresponding to one of the zeroth-order Q-fold degenerate eigenstate) by some linear combination

$$z_m^{(0)} = \sum_{j=1}^{Q} a_{mj} u_j \tag{A2-12a}$$

where the a_{mj} will also have to be determined by the perturbation calculation. We still, of course, take

$$\psi_m^{(0)} = \gamma_m (= \gamma_1 = \gamma_2 = \ldots = \gamma_Q) \tag{A2-12b}$$

Furthermore, pursuing our earlier remarks, we expand $z_m^{(1)}$ and $z_m^{(2)}$ in terms of the $\{u_j\}$. However, since we have already introduced in the zeroth-order term, those u_j which correspond to the unperturbed state, we argue[(9),(10)] that we can omit such

---

*We will focus our attention, for notational convenience, on these first Q eigenstates, where Q is the order of degeneracy associated with the first eigenvalue. It will be trivially obvious to the reader that these considerations apply, with minor notational changes, to calculating perturbations for all of the eigenstates.

terms in expanding $z_m^{(1)}$ and $z_m^{(2)}$, i.e., we write

$$z_m^{(1)} = \sum_{j=Q+1}^{M} b_{mj}\, u_j \tag{A2-13}$$

$$z_m^{(2)} = \sum_{j=Q+1}^{M} c_{mj}\, u_j \tag{A2-14}$$

(It turns out that if one attempts to keep terms in $z_m^{(1)}$ and $z_m^{(2)}$ relating to the zeroth-order eigenstate, the associated coefficients are basically undetermined by the perturbation calculation. One can argue that their value is then supplied by a normalization of the perturbed eigenvector; and this shows them to be always higher-order terms which never enter the calculation at the order being computed. Although there are more sophisticated approaches which over-all appear to deal more precisely with this problem, for the low order to which the perturbation corrections would be used in our case, the more conventional approach being presented appears quite adequate and purposeful.) Further, using Eq. (A2-5a), it now follows that

$$\Gamma^{(0)} z_m^{(1)} = \sum_{j=Q+1}^{M} b_{mj}\, \gamma_j\, u_j \tag{A2-15}$$

$$\Gamma^{(0)} z_m^{(2)} = \sum_{j=Q+1}^{M} c_{mj}\, \gamma_j\, u_j \tag{A2-16}$$

Thus, we can now rewrite Eqs. (A2-10) and (A2-11) as follows:

$$\sum_{j=Q+1}^{M} b_{mj}\, \gamma_j\, u_j + \sum_{j=1}^{Q} a_{mj}\, \Gamma'\, u_j = \sum_{j=Q+1}^{M} b_{mj}\, \gamma_m\, u_j + \sum_{j=1}^{Q} a_{mj}\, \psi_m^{(1)}\, u_j \tag{A2-17}$$

$$\sum_{j=Q+1}^{M} c_{mj}\,\gamma_j\,u_j + \sum_{j=Q+1}^{M} b_{mj}\,\Gamma'\,u_j = \sum_{j=Q+1}^{M} c_{mj}\,\gamma_m\,u_j + \sum_{j=Q+1}^{M} b_{mj}\,\psi_m^{(1)}\,u_j$$

$$+ \sum_{j=1}^{Q} a_{mj}\,\psi_m^{(2)}\,u_j \qquad \text{(A2-18)}$$

Finally, we now apply the orthonormality of the set $\{u_j\}$, i.e.,

$$u_k^{+}\,u_j = \delta_{kj} \qquad \text{(A2-19)}$$

It is also convenient to note, by using Eq. (A2-6b), that

$$u_k^{+}\,\Gamma'\,u_j = \Gamma'_{kj} \qquad \text{(A2-20)}$$

We can thus obtain from Eqs. (A2-17) and (A2-18) (recalling also that $\gamma_1 = \gamma_2 = \ldots = \gamma_Q$),

First Order:

$$k \le Q \qquad \sum_{j=1}^{Q} a_{mj}\,\Gamma'_{kj} = a_{mk}\psi_m^{(1)} \qquad \text{(A2-21)}$$

$$k > Q \qquad b_{mk}\gamma_k + \sum_{j=1}^{Q} a_{mj}\Gamma'_{kj} = b_{mk}\gamma_m \qquad \text{(A2-22)}$$

Second Order:

$$k \le Q \qquad \sum_{j=Q+1}^{M} b_{mj}\,\Gamma'_{kj}\,a_{mk}\psi_m^{(2)} \qquad \text{(A2-23)}$$

$$k > Q \qquad c_{mk}\gamma_k + \sum_{j=Q+1}^{M} b_{mj}\Gamma'_{kj} = c_{mk}\gamma_m + b_{mk}\psi_m^{(1)} \qquad \text{(A2-24)}$$

Eq. (A2-21) may be regarded as a set of Q homogeneous linear algebraic equations in the variable a_{mj}:

$$\sum_{j=1}^{Q} a_{mj}\left(\Gamma'_{kj} - \delta_{kj}\,\psi_m^{(1)}\right) = 0, \qquad k = 1, \ldots , Q \qquad \text{(A2-25)}$$

Non-trivial solutions exist only for values of $\psi_m^{(1)}$ such that

$$\det \left(\Gamma'_{kj} - \delta_{kj}\, \psi_m^{(1)} \right)_Q = 0 \qquad \text{(A2-26)}$$

where the subscript Q is written to indicate that it is a particular submatrix of Γ' which is involved in this calculation. The Q solutions for $\psi_m^{(1)}$ represent the corresponding first-order correction factors for the initially Q-fold degenerate state. If all the values for $\psi_m^{(1)}$ are unequal, the perturbed state is non-degenerate; if some are still equal, these comprise a residually degenerate state, at least to first order. It is interesting to note that for a non-degenerate zeroth-order state (Q = 1 in Eq. A2-26), we have directly

$$\psi_m^{(1)} = \Gamma'_{mm} = 0 \qquad \text{(A2-27)}$$

(we here write m in general, rather than m = 1 as would be specifically required in Eq. (A2-26) by our predicate that we are examining the perturbations of the <u>first</u>, <u>Q-fold</u> degenerate, state, with $m \leq Q$).

For a given solution for $\psi_m^{(1)}$ in Eq. (A2-26), one may solve Eqs. (A2-25) for the corresponding set $\{a_{mj}\}$, with one of the a_{mj} left undetermined. If there is a residual Q'-fold degeneracy, there will be a corresponding set of Q' undetermined coefficients; the Q' additional coefficients can be determined by carrying the computation to a higher order (if this removes the degeneracy), or else constructing an arbitrary mutually orthogonal set, as is normally done in connection with a degenerate eigenstate. In any case, there is always one coefficient left unspecified, which can be fixed by a normalizing requirement, e.g., that the eigenvector $z_m^{(0)}$ defined in Eq. (A2-12) be normalized to unit length,

$$\sum_{j=1}^{Q} \left| a_{mj} \right|^2 = 1 \qquad \text{(A2-28)}$$

We can next determine the b_{mk} $(k > Q)$ from Eq. (A2-22),

$$b_{mk} = \frac{1}{\gamma_m - \gamma_k} \sum_{j=1}^{Q} a_{mj} \Gamma'_{kj} \qquad \text{(A2-29)}$$

Thus far, we have discussed only the first order perturbation. We can now return to the second order. From Eq. (A2-23), if the m^{th} state is one for which the degeneracy was removed in the first order (hence the a_{mj} determined)

$$\psi_m^{(2)} = \frac{1}{a_{mk}} \sum_{j=Q+1}^{M} b_{mj} \Gamma'_{kj}$$

(any $k \leq Q$ may be used here). If the m^{th} state is still, in first-order, one of a Q'-fold degenerate system, we have to instead employ the solution for the $b_{m,}$ in terms of the Q' undetermined a_{mj}, as given by Eq. (A2-29); we substitute this into the appropriate Q' equations of Eq. (A2-23) and end up basically with a set of Q' homogeneous linear algebraic equations in Q' variables. The requirement that the determinant of this Q'-fold system vanish, for non-trivial solutions to exist, then gives the values for $\psi_m^{(2)}$; and then in turn the equations yield the corresponding sets of a_{mj}, with one undetermined and to be fixed by normalization. If there is residual degeneracy in this order, it must be handled just as commented upon earlier (either look for higher order terms, or settle for the degeneracy and construct suitable a_{mj} accordingly).

The determination of the c_{mk} from Eq. (A2-24) is then straightforward. By way of further concluding our results, we note that wherever in our perturbation formulas a factor Γ'_{jk} appears, a factor ϵ appears in conjunction with it. Thus, we see that exactly the forms in Eq. (A2-8b) always enter in. Thus, for instance, we note that in the first order perturbation,

$$z_m = z_m^{(0)} + \epsilon z_m^{(1)}$$

$$= \sum_{j=1}^{Q} a_{mj} u_j + \epsilon \sum_{j=Q+1}^{M} \left[\frac{1}{\gamma_m - \gamma_j} \sum_{p=1}^{Q} a_{mp} \Gamma'_{jp} \right] u_j$$

Or

$$z_m = \sum_{j=1}^{Q} a_{mj} u_j + \sum_{j=Q+1}^{M} \left[\frac{1}{\gamma_m - \gamma_j} \sum_{p=1}^{Q} a_{mp} q_j^* \beta_{jp} q_p \right] \qquad \text{(A2-30)}$$

Similar remarks hold for the eigenvalue, and into all higher-order perturbations.

Finally, we again call to attention that we have basically described the construction of one perturbed eigenstate, or of one member of a perturbed degenerate eigenstate. The calculation must be carried in turn out for each of the total of M eigenstates associated with the Γ-matrix.

APPENDIX 3: OPTIMIZATION OF A TYPICAL ANGLE DIVERSITY ARRAY

a) Radiation Efficiencies

We will consider as a typical angle diversity receiving array that pictured in Figure 5. There is a central beam (Beam 3) and four symmetrically placed side beams (Beams 1, 2, 4, 5). On grounds of physical symmetry, we will assume the latter four are all *a priori* assigned equal radiation efficiencies,

$$q_1 = q_2 = q_4 = q_5 = q \tag{A3-1a}$$

while the center beam has some different efficiency, by a ratio, K,

$$q_3 = Kq \tag{A3-1b}$$

Furthermore, we assume all the beams to be excited by currents in a common aperture, with only phasing across the aperture to produce the different beam pointing directions. In this case, we can take all the beam patterns, $\vec{R}_j(\vec{\Theta})$, to be real-valued, and all identical in form except for angle of pointing. We also assume that all have one common linear polarization. In this case, by the symmetry, we can write

$$\begin{aligned} \beta_{11} &= \beta_{22} = \beta_{33} = \beta_{44} = \beta_{55} = 1 \\ \beta_{12} &= \beta_{25} = \beta_{45} = \beta_{14} = \alpha_1 \\ \beta_{13} &= \beta_{23} = \beta_{43} = \beta_{53} = \alpha_2 \\ \beta_{15} &= \beta_{24} = \alpha_3 \end{aligned} \tag{A3-2}$$

For the numerical example, we will consider a case where

$$1 > \alpha_2 > \alpha_1 > \alpha_3 > 0 \tag{A3-3}$$

On physical grounds, such an inequality will usually hold for the absolute values of the α_j. However, as noted earlier in Appendix 1 (Figure 3), some of the α_j may be negative, and indeed, the inequality may not always hold, even with the configuration assumed. However, it would more usually be the case than not with this configuration, and fortunately, the validity of Eq. (A3-3) for our numerical values serves well to simplify the later arithmetic in our illustrative example.

Then, along with Eqs. (A3-1), we have

$$\Gamma = \begin{pmatrix} |q|^2 & \alpha_1|q|^2 & \alpha_2K|q|^2 & \alpha_1|q|^2 & \alpha_3|q|^2 \\ \alpha_1|q|^2 & |q|^2 & \alpha_2K|q|^2 & \alpha_3|q|^2 & \alpha_1|q|^2 \\ \alpha_2K^*|q|^2 & \alpha_2K^*|q|^2 & |K|^2|q|^2 & \alpha_2K^*|q|^2 & \alpha_2K^*|q|^2 \\ \alpha_1|q|^2 & \alpha_3|q|^2 & \alpha_2K|q|^2 & |q|^2 & \alpha_1|q|^2 \\ \alpha_3|q|^2 & \alpha_1|q|^2 & \alpha_2K|q|^2 & \alpha_1|q|^2 & |q|^2 \end{pmatrix} \quad \text{(A3-4)}$$

The eigenvalues, γ, of Γ are determined by solving $\det(\Gamma - \gamma I) = 0$

$$0 = \begin{pmatrix} |q|^2-\gamma & \alpha_1|q|^2 & \alpha_2K|q|^2 & \alpha_1|q|^2 & \alpha_3|q|^2 \\ \alpha_1|q|^2 & |q|^2-\gamma & \alpha_2K|q|^2 & \alpha_3|q|^2 & \alpha_1|q|^2 \\ \alpha_2K^*|q|^2 & \alpha_2K^*|q|^2 & |K|^2|q|^2-\gamma & \alpha_2K^*|q|^2 & \alpha_2K^*|q|^2 \\ \alpha_1|q|^2 & \alpha_3|q|^2 & \alpha_2K|q|^2 & |q|^2-\gamma & \alpha_1|q|^2 \\ \alpha_3|q|^2 & \alpha_1|q|^2 & \alpha_2K|q|^2 & \alpha_1|q|^2 & |q|^2-\gamma \end{pmatrix}$$

(A3-5)

This may be simplified by factoring K ($K \neq 0$) from the third column, and K^* from the third row; and also factoring $|q|^2$ from every row. Defining

$$x = \frac{\gamma}{|q|^2} \tag{A3-6}$$

we then have

$$0 = \begin{vmatrix} 1-x & \alpha_1 & \alpha_2 & \alpha_1 & \alpha_3 \\ \alpha_1 & 1-x & \alpha_2 & \alpha_3 & \alpha_1 \\ \alpha_2 & \alpha_2 & 1 - \frac{x}{|K|^2} & \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_3 & \alpha_2 & 1-x & \alpha_1 \\ \alpha_3 & \alpha_1 & \alpha_2 & \alpha_1 & 1-x \end{vmatrix} \tag{A3-7}$$

We note at this point that K only enters through $|K|^2$ and hence with no loss in generality, we may hereafter regard K as real. Next, Eq. (A3-7) can be simplified by subtracting the last column from the first, and the fourth from the second:

$$0 = \begin{vmatrix} 1-\alpha_3-x & 0 & \alpha_2 & \alpha_1 & \alpha_3 \\ 0 & 1-\alpha_3-x & \alpha_2 & \alpha_3 & \alpha_1 \\ 0 & 0 & 1 - \frac{x}{K^2} & \alpha_2 & \alpha_2 \\ 0 & \alpha_3+x-1 & \alpha_2 & 1-x & \alpha_1 \\ \alpha_3+x-1 & 0 & \alpha_2 & \alpha_1 & 1-x \end{vmatrix} \tag{A3-8}$$

Next, adding the first row to the last, and the second to the fourth:

$$0 = \begin{vmatrix} 1-\alpha_3-x & 0 & \alpha_2 & \alpha_1 & \alpha_3 \\ 0 & 1-\alpha_3-x & \alpha_2 & \alpha_3 & \alpha_1 \\ 0 & 0 & 1-\frac{x}{K^2} & \alpha_2 & \alpha_2 \\ 0 & 0 & 2\alpha_2 & 1+\alpha_3-x & 2\alpha_1 \\ 0 & 0 & 2\alpha_2 & 2\alpha_1 & 1+\alpha_3-x \end{vmatrix}$$

Hence, we can immediately reduce the equation to

$$0 = (1-\alpha_3-x)^2 \begin{vmatrix} 1-\frac{x}{K^2} & \alpha_2 & \alpha_2 \\ 2\alpha_2 & 1+\alpha_3-x & 2\alpha_1 \\ 2\alpha_2 & 2\alpha_1 & 1+\alpha_3-x \end{vmatrix} \tag{A3-9}$$

If we now subtract the third row from the second, and then add the new second column to the third,

$$0 = (1-\alpha_3-x)^2 \begin{vmatrix} 1-\frac{x}{K^2} & \alpha_2 & 2\alpha_2 \\ 0 & 1+\alpha_3-2\alpha_1-x & 0 \\ 2\alpha_2 & 2\alpha_1 & 1+2\alpha_1+\alpha_3-x \end{vmatrix}$$

$$= (1-\alpha_3-x)^2(1+\alpha_3-2\alpha_1-x) \begin{vmatrix} 1-\frac{x}{K^2} & 2\alpha_2 \\ 2\alpha_2 & 1+2\alpha_1+\alpha_3-x \end{vmatrix} \tag{A3-10}$$

Thus the five roots are:

$$x_1 = x_2 = 1-\alpha_3 \quad \text{(double root)}$$

$$x_3 = 1+\alpha_3-2\alpha_1$$

and x_4, x_5 are the solutions to

$$\left(1 - \frac{x}{K^2}\right)(1+2\alpha_1+\alpha_3-x) - 4\alpha_2^2 = 0$$

$$x^2-x(K^2+1+2\alpha_1+\alpha_3) + K^2(1+2\alpha_1+\alpha_3) - 4K^2\alpha_2^2 = 0 \qquad \text{(A3-12a)}$$

$$\left.\begin{matrix} x_4 \\ x_5 \end{matrix}\right\} = \frac{(K^2+1+2\alpha_1+\alpha_3) \pm \sqrt{\left[K^2+1+2\alpha_1+\alpha_3\right]^2 + 4K^2\left[4\alpha_2^2 - (1+2\alpha_1+\alpha_3)\right]}}{2}$$

which can alternatively be written

$$\left.\begin{matrix} x_4 \\ x_5 \end{matrix}\right\} = \frac{(K^2+1+2\alpha_1+\alpha_3) \pm \sqrt{\left[K^2 - (1+2\alpha_1+\alpha_3)\right]^2 + 16K^2\alpha_2^2}}{2} \qquad \text{(A3-12b)}$$

We note that in Eq. (A3-12) the larger eigenvalue is x_4 and the smaller is x_5. From the latter, since on physical grounds it cannot be negative, we see that we must have

$$4\alpha_2^2 - (1+2\alpha_1+\alpha_3) < 0$$

$$\alpha_2^2 < \frac{1+2\alpha_1+\alpha_3}{4} \qquad \text{(A3-13)}$$

Although not obvious by inspection, this latter inequality probably follows from the basic assumptions of our configuration.

Finally, now, from Eq. (A3-12b), we note that

$$x_4 > \frac{(K^2+1+2\alpha_1+\alpha_3) + \left|K^2 - (1+2\alpha_1+\alpha_3)\right|}{2}$$

Or, for

$$\begin{cases} K^2 < 1+2\alpha_1+\alpha_3 \quad , & x_4 > 1+2\alpha_1+\alpha_3 \\ K^2 > 1+2\alpha_1+\alpha_3 \quad , & x_4 > K^2 > 1+2\alpha_1+\alpha_3 \end{cases}$$

In either case then,

$$x_4 > 1+2\alpha_1+\alpha_3 > x_1 \text{ or } x_3 \tag{A3-14}$$

and hence x_4 is always the largest eigenvalue.

Recalling Eq. (A3-6) and Eq. (A3-9), we thus have immediately

$$(\gamma_k)_{max} = |q|^2 x_4 \leq 1$$

$$|q|^2 \leq \frac{1}{x_4}$$

and, with rationalization of the resulting fraction,

$$|q|^2 \leq \frac{(K^2+1+2\alpha_1+\alpha_3) - \sqrt{(K^2+1+2\alpha_1+\alpha_3)^2 - 4K^2\left[(1+2\alpha_1+\alpha_3) - 4\alpha_2^2\right]}}{2K^2\left[1+2\alpha_1+\alpha_3 - 4\alpha_2^2\right]} \tag{A3-15}$$

We may note that for Beam 3, the corresponding radiation efficiency is

$$K^2|q|^2 \leq \frac{(K^2+1+2\alpha_1+\alpha_3) - \sqrt{(K^2+1+2\alpha_1+\alpha_3)^2 - 4K^2(1+2\alpha_1+\alpha_3-4\alpha_2^2)}}{2(1+2\alpha_1+\alpha_3-4\alpha_2^2)} \tag{A3-16}$$

(It is incidentally readily verified that $K^2 \left|q\right|^2 \leq 1$ also, as is required physically.)

Let us now assume the equality sign in Eqs. (A3-15) and (A3-16), i.e., that whatever the value of $|K|$, we have a lossless antenna system so designed that there are no cross-couplings or losses other than fundamentally inherent and implied by the beam overlaps and the conservation of energy limitations.

b) Optimization of Diversity Operation

We now refer to the optimization problem discussed in Section 8. We will assume that the matrix, G, of beam correlation factors, G_{kj} (defined in Eq. 50) has the same kind of symmetries that the β-matrix had. This is basically the assumption that $g(\theta)$ in Eq. (50) is circularly symmetric, and that Beam 3 is pointed at its center of symmetry. This assumption is not a particularly accurate description of the tropospheric scatter fields, which display a different scale of variations in the vertical than in the horizontal directions. However, since our purposes here are purely illustrative, we believe the simplification in resulting mathematics justifies the symmetry assumption on $g(\theta)$. The point is that with this assumed symmetry, we can immediately note symmetries in G_{jk} which are almost, but not completely analogous to those in Eq. (A3-2); we write these as

$$
\begin{aligned}
G_{33} &= h_o'' \\
G_{11} = G_{22} = G_{44} = G_{55} &= h_o' \\
G_{12} = G_{25} = G_{45} = G_{14} &= h_1' \\
G_{13} = G_{23} = G_{43} = G_{53} &= h_2' \\
G_{15} = G_{24} &= h_3'
\end{aligned}
\qquad \text{(A3-17)}
$$

Hence, we can determine the eigenvalues, $\left\{ \ell_j \right\}$, for the L-matrix (Eq. 51) in almost direct analogy to those determined for Γ. Thus, if we define

$$h_1 = \frac{h_1'}{h_o'}$$

$$h_2 = \frac{h_2'}{h_o''}$$

$$h_3 = \frac{h_3'}{h_o'} \tag{A3-18a}$$

$$\left| K' \right|^2 = \left| K \right|^2 \frac{h_o''}{h_o'}$$

and

$$y_i = \frac{\ell_i}{h_o' \left| q \right|^2} \tag{A3-18b}$$

we obtain as the solutions for the eigenvalues

$$\ell_1 = \ell_2 = h_o' \left| q \right|^2 y_1 = h_o' \left| q \right|^2 y_2 = h_o' \left| q \right|^2 (1-h_3) \tag{A3-19a}$$

$$\ell_3 = h_o' \left| q \right|^2 y_3 = h_o' \left| q \right|^2 \; (1+h_3-2h_1) \tag{A3-19b}$$

$$\left.\begin{matrix} \ell_4 \\ \ell_5 \end{matrix}\right\} = h_o' \left| q \right|^2 \left.\begin{matrix} y_4 \\ y_5 \end{matrix}\right\}$$

$$= h_o' \left| q \right|^2 \; \frac{(K'^2+1+2h_1+h_3) \pm \sqrt{(K'^2+1+2h_1+h_3)^2+4K'^2 \left[4h_2^2-(1+2h_1+h_3)\right]}}{2} \tag{A3-19c}$$

In these, $|q|^2$ is now assumed to be given by the equality sign in Eq. (A3-15) (in terms of K, not yet specified). Furthermore, the error rate function F(z) to be minimized for optimum performance (Eq. 47), is

$$F(z) = \frac{1}{2}\,\frac{1}{\det(I+zL)} \tag{A3-20}$$

(z will be further specified below.) With L Hermitian, one knows that a diagonalizing unitary matrix exists for L, say T, such that

$$T^{+}T = I \tag{A3-21}$$

and

$$T^{+}LT = \ell$$

where ℓ is the diagonal matrix

$$\ell = \begin{pmatrix} \ell_1 & & & 0 \\ & \ell_2 & & \\ & & \ddots & \\ 0 & & & \ell_5 \end{pmatrix} \tag{A3-22}$$

But then, since det T = 1,

$$\det(I+zL) = \det(T^{+}IT + zT^{+}LT) = \det(I+z\ell)$$

$$= \prod_{i=1}^{5} (1+z\,\ell_i) \tag{A3-23}$$

Hence,

$$F(z) = \frac{1}{2}\,\frac{1}{\prod_{i=1}^{5} (1+z\,\ell_i)} \tag{A3-24}$$

$z = \frac{1}{2N_o}$, N_o = noise level in individual receiver (same for all receivers) and we have F(z) as an explicit function of K through Eqs. (A3-19) and (A3-15). To maximize the resulting expression algebraically is tedious. Therefore, we may take what may be regarded as a typical set of numbers for α_1, α_2, α_3 (see Appendix 1), and another typical set for h_o'', h_o', h_1', h_2', h_3'. With these, we will plot in Figures 7 and 8 the variation of F(z) as a function of K for various assumed average signal-noise ratios.

The particular numerical values taken are for gaussian beam patterns, in the configuration of Figure 5, where Beam 3 intersects the other beams at their 3db points. From Figures 2 and 3, we then obtain

$$\alpha_1 = 0.49$$

$$\alpha_2 = 0.24$$

$$\alpha_3 = 0.06$$

For the intensity pattern $g(\Theta)$, we assume a form

$$g(\Theta) = \frac{Q_o}{1 + \left(\frac{\Theta}{\Theta_o}\right)^2}$$

where Θ, normalized to the same angle coordinates used in Figures 2 and 3, is in effect the source "spread" normalized to the beam width of our individual beams. The value Q_o determines the actual power level of the received signals. The values obtained for the G matrix are then:

$Q_o = 1$	$Q_o = 2$
$h_o'' = 0.1574\ Q_o$	$h_o'' = 0.2697\ Q_o$
$h_o' = 0.0644\ Q_o$	$h_o' = 0.1832\ Q_o$
$h_1' = 0.0442\ Q_o$	$h_1' = 0.0984\ Q_o$
$h_2' = 0.0833\ Q_o$	$h_2' = 0.1632\ Q_o$
$h_3' = 0.0306\ Q_o$	$h_3' = 0.0524\ Q_o$

The individual ℓ_i will thus be proportional to Q_o. Thus, for each term in Eq. (A3-24),

$$z\,\ell_i = \frac{Q_o}{2N_o}\,\ell_i'$$

where ℓ_i' is the value of the eigenvalues if $Q_o = 1$ is taken in defining the elements of the G-matrix. Clearly, furthermore, $h_o'' \frac{Q_o}{2N_o}$ is the signal-noise ratio which would appear in the receiver associated with the central beam if no other beams were present (and $|K^2|\,|q|^2 = 1$ were involved). Thus, let us redefine z as the latter signal-noise ratio, and rewrite Eq. (A3-25) in the form

$$z = h_o'' \frac{Q_o}{2N_o} \tag{A3-25a}$$

$$F(z) = \frac{1}{2}\,\frac{1}{\prod_{i=1}^{5}\left[1 + \left(\frac{z}{h_o''}\right)\ell_i'\right]} \tag{A3-25b}$$

In Figures 6-8 have been plotted the radiation efficiencies and $F(z)$ as functions of $|K|^2$. The latter variable has been graphed logarithmically, to bring out the essential features. In almost all cases, the values plotted for $|K|^2 > 0.1$ are relatively close to those for $|K|^2 = 0$ (the latter have been indicated via arrows). In Figure 6 are given the radiation efficiencies, $|q|^2$, for the four outer beams, and $|K|^2|q|^2$ for the central beam. It is clear that for $|K|^2 \longrightarrow 0$, the central beam is effectively omitted, and the residual limitation on $|q|^2$ is the interaction only among the four outer beams. Likewise, for $|K|^2$ very large, $|K|^2\,|q|^2 \longrightarrow 1$ while $|q|^2 \longrightarrow 0$, and in effect only the central beam is present. In Figures 7 and 8 are given plots of the error rate, $\frac{1}{2}F(z)$, for selected values of z (0, 5, 10, 15, and 20 db), and for $\Theta_o = 1$ and $\Theta_o = 2$, respectively. The ordinate is broken, since for the range of z taken, the error rates vary over several decades, whereas it seems desirable to plot all the curves on one graph.

The most striking observations from Figures 7 and 8 are that there is an optimum value for $|K|^2$, generally around $|K|^2 = 0.7$, and somewhat independent of z or Θ_o; that the error rate curves have a broad minimum around the optimum, so that the choice is not critical; that, indeed, for the configuration assumed, and the signal-noise ratios examined, the variation of error rate is by no more than a factor of 2 over the range $0.1 \leq K^2 \leq 1$. Changes in error rate by a factor of 2 are equivalent, due to the exponential dependence of error rate on signal-noise ratio, to very minor changes in signal-noise ratio. Thus one may especially notice that in the configuration examined, the optimum presence of the center beam (optimum $|K|^2$) is very little better than the complete <u>absence</u> of the center beam ($|K|^2 = 0$). That is, the added losses (due to cross-coupling) in the power received with the outer beams, when the center beam is active, almost completely compensate for the additional diversity advantage of adding this center beam. (Indeed for the $\Theta_o = 2$ case, the optimum $|K|^2$ for small signal-noise ratio is in fact $|K|^2 = 0$.)

It is known that for sufficiently low error rates (sufficiently high signal-noise ratio), the added diversity advantage of having five rather than four beams must appear. This is indicated by the decreasing width of the minimum as signal-noise ratio increases. However, for our example, the error-rate at which this diversity advantage appears is apparently far below those which appear in our curves, which in turn are typical of operational requirements.

One may also question whether the beam overlap factors used here are realistic. We have not, in all this, specifically derived the related cross-couplings between the feed-lines. In an actual design, as mentioned earlier, the design engineer might attempt to modify the feed structure to reduce such cross-couplings, in a way which would not alter the main lobe pattern but might drastically alter the side-lobes and hence the overlap factors. (One may also question, on the other hand, whether this is really ever done in designing an array for receiving only; the penalties are obvious for a transmitting array, but it is not apparent that they have ever been regarded seriously for receiving-only arrays.)

At any rate, this example should be taken only as an illustration of the kind of effects which have been imputed from the mathematics, and as an indication of the kind of detailed investigation which should be undertaken in connection with any angle-diversity designs.

ON CROSS-COUPLING IN MULTIPLE BEAM ANTENNAS

PART II CHARACTERISTICS OF THE CROSS-COUPLING MATRICES

by

S. Stein and J.E. Storer

ABSTRACT

The analysis in Part I has been extended to the characterization of the scattering matrix for a multiple-beam antenna system. The constraints implied by beam overlap, and the principle of conservation of energy, are derived. The more specific properties are given for antennas further characterized as lossless, reciprocal systems. Some directions for further useful research are pointed out.

1. Summary of Prior Results

In Part I*, the principle of conservation of energy was applied to a multiple beam antenna (M beams). An Hermitian positive semi-definite MxM matrix, β, of beam-overlap factors was defined, and a set of M beam excitation factors, $\{q_k\}$, for which $|q_k|^2$ represent the radiation efficiency for the k^{th} beam. Another MxM Hermitian positive semi-definite matrix, Γ, was defined, with its elements given by

$$\Gamma_{kj} = q_k^* \beta_{kj} q_j \tag{1}$$

Furthermore, the antenna system, considered as a junction among the M feed-lines (each excites exactly one of the beams), is characterized by an MxM "scattering-matrix", S, whose elements S_{kj} represent the mutual coupling coefficients (and self-reflection coefficients) among the set of feedlines. For a given set of incident waves in the several feed-lines, with amplitudes $\{x_k\}$ characterized as a column vector, x, the set of reflected waves is denoted by another set $\{y_k\}$, or an equivalent column vector, y,

$$y = Sx \tag{2}$$

The specific result obtained from applying the principle of conservation of energy involves the eigenvalues and eigenvectors of Γ. Since Γ is Hermitian, and positive semi-definite, its M eigenvalues are known to be positive real or zero; we denote by γ the diagonal matrix (i.e., matrix with non-vanishing elements only on the main diagonal) whose diagonal elements are these eigenvalues of Γ,

$$\gamma = \begin{pmatrix} \gamma_1 & & & 0 \\ & \gamma_2 & & \\ & & \ddots & \\ 0 & & & \gamma_M \end{pmatrix} \tag{3}$$

*S. Stein, "On Cross-Coupling in Multiple-Beam Antennas, Part I, Radiation Efficiencies (With Applications to Angle Diversity)," ARM No. 237, 3 March 1961.

The γ_k, $\gamma_k \geq 0$ are the aforementioned eigenvalues. Furthermore, the eigenvectors of Γ form a mutually orthogonal set (or, for a degenerate eigenvalue, can be constructed as such), and can further be normalized to unit length. Within a lack of uniqueness discussed in Appendix 1, one can define a unitary matrix, U, whose columns are the respective orthonormalized eigenvectors of Γ, and which has the properties*

$$U^+U = I \qquad \text{(I is the identity matrix, } I_{ij} = \delta_{ij}\text{)}$$

$$U^+\Gamma U = \gamma \tag{4}$$

$$\Gamma = U\gamma U^+$$

The fundamental equation derived in Part I (Part I, Eq. 36) was the inequality

$$z^+(I - \gamma)z \geq z^+(U^+S^+SU)z \tag{5}$$

where the equality sign holds if and only if a lossless antenna system is being considered. The immediate deduction from Eq. 5 was that all the eigenvalues of Γ are restricted by the inequalities

$$1 \geq \gamma_k \geq 0 \tag{6}$$

Most of Part I was devoted to exploring the consequences of Eq. 6, with respect to limitations imposed by beam-overlaps (by the form of β) upon the achievable radiation efficiencies. Once the latter are chosen, consistent with Eq. 6, the γ_k are completely specified within the assumed beam patterns.

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*Again we use the notations, for any complex matrix A,

$$A^* = \text{complex conjugate of A; } (A^*)_{ij} = A^*_{ij}$$

$$A^T = \text{transpose of A; } (A^T)_{ij} = A_{ji}$$

$$A^+ = \text{conjugate transpose of A; } (A^+)_{ij} = A^*_{ji}$$

In this report, we will explore the further implications of Eqs. 5 and 6, with respect to characterizations of the scattering matrix. We will provide a rather complete characterization for the very important, most common case of a lossless reciprocal antenna system, which includes all antenna systems comprised of essentially passive lossless elements. For more general systems, we will only be able to indicate a partial characterization, which includes all the implications of conservation of energy and the beam specifications, but apparently always requires other constraints (such as losslessness, reciprocity) to complete the description of the system.

2. General Solution for S

In Eq. 5 (along with Eq. 6), we recognize the I-γ is a diagonal matrix with elements positive real or zero. We can thus define a diagonal matrix, Λ, whose elements are the set of positive real or zero numbers

$$\lambda_k = \sqrt{1 - \gamma_k} \geq 0 \qquad k = 1, \ldots, M \tag{7}$$

$$\Lambda_{kj} = \lambda_k \delta_{kj} \tag{8a}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_M \end{pmatrix} \tag{8b}$$

We note

$$\Lambda = \Lambda^T = \Lambda^* = \Lambda^+ \tag{8c}$$

We can then write

$$I - \gamma = \Lambda^2 \tag{9}$$

and Eq. 5 becomes

$$z^+ \Lambda^2 z \geq z^+ (U^+ S^+ S U) z \tag{10}$$

Now in Part I, it was shown that with all other relevant parameters fixed, the maximum radiation efficiencies for all the beams are attained when the largest of the γ_k is equal to unity (the largest value compatible with Eq. 6). Both the diagonal matrices (I-γ) and Λ would then have some of their diagonal elements zero (as many such elements as the order of degeneracy of the largest eigenvalue). To avoid the mathematical complications necessitated by accounting for this effect, we will assume in this section that none of the λ_k vanish (all γ_k are less than unity). On this basis, Λ has a well-defined inverse Λ^{-1}, which we will use. However this inverse will be seen to be absent from the final result, and one will be able to argue then that the result applies as well to the limiting case where some of the λ_k do vanish. In Appendix 2, we further indicate by paralleling the derivation in this section, a more rigorous demonstration of this assertion.

Now, assuming Λ^{-1} exists, let us define a new matrix

$$T = SU\Lambda^{-1} \tag{11a}$$

$$S = T\Lambda U^{+} \tag{11b}$$

so that Eq. 10 becomes

$$z^{+}\Lambda^{2}z \geq z^{+}\Lambda\ T^{+}T\Lambda z \tag{12}$$

Defining a new (equally arbitrary) vector

$$\zeta = \Lambda z \tag{13}$$

we can further write the inequality as

$$\zeta^{+}\zeta \geq \zeta^{+}T^{+}T\zeta \tag{14}$$

Clearly, the matrix $T^{+}T$ is Hermitian, and positive definite, since $S^{+}S$ is positive definite (shown in Part I). Thus it can always be written in the form

$$T^{+}T = P^{+}\mu^{2}P \tag{15a}$$

where P is an appropriate unitary matrix

$$P^{+}P = I \tag{15b}$$

and μ^2 is a diagonal matrix whose diagonal elements, μ_k^2, are positive real or zero. The requirement imposed by Eq. 14 is then readily observed (see, e.g., Appendix 2) to be simply that

$$\mu_k^2 \leq 1 \qquad \text{for all } k \tag{16}$$

Since the latter statement is the totality of information contained in the conservation of energy equation, Eq. 14, we see that the class of all permissible solutions for $T^{+}T$ is hence given by Eq. 15, where P is any unitary matrix, and μ^2 any diagonal matrix satisfying Eq. 16.

It immediately follows by "factoring" Eq. 15a that within this class of solutions, T itself has as its most general form

$$T = W\mu P \tag{17}$$

where W is again any unitary matrix

$$W^{+}W = I \tag{18}$$

Using Eq. 11, the corresponding general class of solutions for S, consistent with conservation of energy, is just

$$S = W\mu P\Lambda U^{+} \tag{19}$$

In the general class of solutions just given, only the diagonal matrix Λ is completely specified (by our choice of the multiple beams, and the choice of radiation efficiencies), and the matrix U partially specified (see Appendix 1). We can expect, and will discuss below, that the arbitrariness of choice of the unitary matrices P and W has physical implications concerning certain freedoms of choice in designing the antenna system to drive the specified set of beams. The inequality which specifies μ, however, is a source of difficulty. In the lossless case, of course, Eq. 16 is to be taken with the equality sign, and we will see below that this greatly reduces the generality of possible forms for S.

In the more general lossy case, the inequality signs in our equations, Eq. 5 for example, could similarly be replaced by equality signs if a term were added which accounted for all losses within the antenna system (considered as a junction), i.e., Eq. 5 rewritten as

$$z^{+}(I - \gamma)z = z^{+}(U^{+}S^{+}SU)z + L \tag{20}$$

where L is a term accounting for all the losses. If, in fact, such losses were all ohmic, L would be a Hermitian quadratic form in z,

$$L = z^{+}(U^{+}L_{o}U)z \tag{21}$$

(where L_o is the Hermitian matrix of the quadratic form giving the losses, when the variable is the set of incident waves in the feedlines). If L, or L_o, were known, one could presumably make corresponding specific statements about μ, much like the ones to be made below in the lossless case. If the losses were due to specific "pads" inserted into the network, one might be readily able to describe such a matrix. under more general lossy conditions, however, it is not apparent that enough is generally known about the quadratic form to make use of it. Results in the remainder of this paper will be confined to the lossless case.

Other specializations can be made, not related to the lossless vs. lossy issue, for example the assumption of reciprocity. Combined with losslessness, below, this yields a particularly interestingdelineation on the form of S. The characterization assuming reciprocity only, however, does not appear to be useful and again will not be cited explicitly. As we comment in our conclusions, there thus seem to be a number of interesting problems yet to be explored in this general direction of system constraints. These we necessarily regard as beyond the scope of our currently achievable results presented in this paper.

4. Lossless Reciprocal Systems

By far, the widest class of antenna systems can be regarded as possessing both the properties of reciprocity and losslessness.

For a lossless system, the equality sign holds in Eq. 16, and we have

$$\mu^2 = I \tag{22}$$

Since $\mu = \mu^+$, μ itself can be replaced by any unitary matrix; the resulting product $W\mu P$ in Eq. 17 becomes simply the product of three arbitrary unitary matrices, and leads to the simple statement

$$T = \text{any unitary matrix} \tag{23}$$

(This result is also apparent by simply setting $\mu^2 = I$ in Eq. 15). Thus, the property of losslessness reduces to a statement

$$S = Q\Lambda U^+ \tag{24a}$$

where Q is any unitary matrix,

$$Q^+Q = I \tag{24b}$$

In order for the network to be also reciprocal, we require

$$S = S^T \tag{25}$$

(note, since S is complex, this is not Hermitian symmetry)
Clearly arbitrary choices of Q will not satisfy this requirement, and hence would generally imply non-reciprocal networks. (Aside from the difficulties in specifying particular kinds of non-reciprocity, if one may use such vague terminology, it was also pointed out in Part I that such systems are basically irrelevant in the important problem of analyzing receiving systems in terms of equivalent transmitting properties.) Now, to satisfy the reciprocity requirement in Eq. 25, the class of solutions in Eq. 24 must also obviously satisfy the relation

$$Q\Lambda U^{+} = (Q\Lambda U^{+})^{T} = U^{*}\Lambda Q^{T} \tag{26}$$

If we define a new matrix, F, by

$$Q = U^{*}F \tag{27}$$

then F must also be unitary,

$$F^{+}F = I \tag{28a}$$

and satisfy, from Eq. 26,

$$F\Lambda = \Lambda F^{T} \tag{28b}$$

In terms of components, since Λ is diagonal, Eq. 28b becomes

$$F_{ij}(\lambda_i - \lambda_j) = 0 \tag{29}$$

I. e., either $\lambda_i = \lambda_j$ or $F_{ij} = 0$. Thus if all the diagonal elements are unequal, F will itself be any <u>diagonal</u> unitary matrix. In the more general case, if some of the λ_i have equal values (corresponding to degenerate eigenvalues of Γ), off-diagonal elements are possible, at intersections of the corresponding rows and columns. If, without loss in generality, we assume the λ_i for the moment to be ordered (grouped), then clearly the non-vanishing elements allowed by Eq. 29 must be clustered such that F will be a partitioned diagonal form,

$$F = \begin{pmatrix} \boxed{F_1} & & & 0 \\ & \boxed{F_2} & & \\ & & \ddots & \\ 0 & & & \boxed{F_n} \end{pmatrix} \tag{30}$$

where the size of the partitions, F_i, is equal to the order of multiplicity of the corresponding eigenvalues. Within this context, Eqs. 28 simply now require that the individual partitions, F_i, be <u>unitary and symmetric</u> matrices (within their dimensionality).

However, it is well known that any symmetric matrix can always be factored, in an infinite number of ways, as the product of some matrix and its transpose, i.e., for each F_i we can write

$$F_i = V_i V_i^{T} \tag{31}$$

Furthermore, to satisfy the unitary property of F_i, it is sufficient to choose only those factorizations, V_i, which are themselves unitary

$$V_i^+ V_i = I \tag{32}$$

(it is readily shown that such V_i exist). Making any such factorization for the F_i, we can then obviously write

$$F = VV^T \tag{33}$$

where V is a diagonal partitioned unitary matrix of form similar to F,

$$V = \begin{pmatrix} \boxed{V_1} & & & 0 \\ & \boxed{V_2} & & \\ & & \ddots & \\ 0 & & & \boxed{V_n} \end{pmatrix} \tag{34}$$

Then S has the form

$$S = Q\Lambda U^+ = U^* F \Lambda U^+ = U^* VV^T \Lambda U^+ \tag{35}$$

But, finally, any diagonal partitioned matrix such as V commutes with Λ (analogous to Eq. A1-8) and thus we can always write this S in the form

$$\begin{aligned} S &= U^* V \Lambda V^T U^+ \\ &= (U^* V)\Lambda(U^* V)^T \end{aligned} \tag{36}$$

But (Appendix 1), with V unitary and partitioned as shown, U^*V is simply U_1^* where U_1 is some other unitary matrix which diagonalizes Γ; i.e., within the lack of uniqueness of the matrix U, our most general solution for S in the lossless reciprocal case has the form

$$S = U^* \Lambda U^+ \tag{37}$$

5. Physical Significance of the Lossless, Reciprocal Characterization

For a given multiple-beam design, and decision on beam radiation efficiencies, the matrix Λ is completely fixed, and the unitary matrices, U, which enter Eq. 37 are fixed to within certain quasi-diagonal forms (Eq. A1-7). It is useful to interpret the meaning of this arbitrariness still inherent in the specification of S via Eq. 37.

We recall, Eq. 2, that S relates the set of reflected waves in the feed-lines to the set of incident waves,

$$y = Sx$$

Using Eq. 37, this can be written

$$y = U^{*}\Lambda U^{+}x$$

Or

$$U^{T}y = \Lambda U^{+}x \tag{38}$$

Let us <u>define</u> "canonical" incident and reflected waves, y' and x' by

$$x' = U^{+}x \quad ; \quad x = Ux'$$
$$y' = U^{T}y \quad ; \quad y = U^{*}y' \tag{39}$$

Then also

$$y' = \Lambda x' \tag{40a}$$

and in components

$$y'_k = \lambda_k x'_k \tag{40b}$$

But what, in fact, are these "canonical waves"? Well, first the unitary matrix U in Eq. 39 may be regarded as a transfer matrix for a lossless "black box", with M ports on each side, such that putting in the $\{x'_k\}$ at one end gives the $\{x_k\}$ at the other (with no reflections), and vice-versa, if the reflections from the antenna $\{y_k\}$ are traveling back towards this box, the set $\{y'_k\}$ emerges without losses. Obviously such a "black box" satisfies all our conservation of energy limitations, and we should not be surprised to find this degree of flexibility in our final equations. (One can dispute whether such a lossless, reflectionless black box is realizable; our equations only say that if realizable, it can be inserted.)

Further, each of the x'_k represents a particular weighted sum of the actual incident waves

$$x'_k = \sum_j U^{+}_{kj}x_j \tag{41}$$

and inversely, given the set $\{x_k'\}$,

$$x_k = \sum_j U_{kj} x_j' \tag{42}$$

The set of excitations, $\{x_k\}$, corresponding to one of the x_j' non-vanishing and all others zero, obviously defines some combination of beams, i.e., some radiation pattern. Thus, each of the x_j' corresponds to exciting some particular radiation pattern, and (referring to Part I[1]) it is readily observed that since the weightings in Eq. 42 are from a unitary matrix which diagonalizes Γ, the set of corresponding "canonical" radiation patterns (beam combinations) are in fact orthogonal to each other. Since they are orthogonal, their "feed-lines" are decoupled; as shown in Eq. 40b, a particular x_j' gives rise only to the corresponding "reflection", y_j' in its own feed-line. The further arbitrariness in U corresponds to the statement that certain "canonical" pattern excitations are equivalent as far as the reflections produced (equal λ_k), and hence as far as efficiency in radiating the energy incident in the feeds. Thus the "canonical" pattern representation is not unique; in particular, even if all the λ_k were unequal, phase shifters in the lines (or equivalently, shift in choice of phase references) would in no manner change the basic energy flow relationships. It is also interesting to note that the eigenvalues γ (of Γ) are directly interpretable in these canonical excitations, in terms of the reflection coefficients, through the relations

$$\lambda_k = \sqrt{1 - \gamma_k}$$

It is not clear to what extent the "canonical" patterns may be of direct physical interest. Presumably, our physical interest lies in the set of beam patterns which we specified at the outset. Although Eq. 42 also can be regarded as describing how to achieve this set by simultaneous excitation at a number of terminals, each of which is decoupled from the other, the totality of waves for a single desired beam involves all terminals, and no obvious engineering advantage ensues. What appears to be more to the point is a study, within some specific systems applications of multiple-beam antennas, of the possibilities for finding a "minimum" S in some sense. The important result is that the character of S is largely fixed by the beam design, and it could be important to an antenna design engineer to understand what aspects of the cross-couplings (elements of S) cannot be changed without altering the beam structure.

6. Need for Further Study

The results of this paper can only be regarded as a first step in characterizing the cross-couplings within multiple-beam antenna systems. It appears that more explicit relationships ought to be determined, probably in relation to specific system applications, in order to arrive at the really desirable point for the antenna design engineer: to determine what limitations he faces a priori in trying to modify an antenna structure so as to reduce cross-couplings without exceeding radiation pattern tolerances specified by his customer. For example, in Part I it was pointed out that much of the cross-coupling appears to be controlled by side-lobe effects, where perhaps more freedom is possible. The important factor would be the detailed nature of the connection, and this we do not claim to have truly elucidated in this initial formulation.

What this paper has achieved, we believe, is to make explicit the basic mathematical structure which can be a starting point for such further research. Over and above such additional research in even the "simple" lossless, reciprocal case, lie further explorations pointed out earlier in connection with such system properties as "lossy", "non-reciprocal", and the like.

APPENDIX 1: Lack of Uniqueness in Unitary Diagonalizing Matrices

Consider an M×M Hermitian matrix Γ. Its eigenvectors form a mutually orthogonal set, or, for a degenerate eigenvalue can be constructed as such (the Gram-Schmidt procedure). Further, the eigenvectors can be normalized to unit length. Thus, if $\Psi^{(k)}$ denotes the k^{th} orthonormalized eigenvector (with components $\Psi_m^{(k)}$, m = 1, . . . , M), we can write the matrix statement

$$\Psi^{(k)+}\Psi^{(j)} = \delta_{kj} \tag{A1-1}$$

A unitary matrix U with the set $\{\Psi^{(k)}\}$ as its columns then has the property of diagonalizing Γ (Eq. 4)

$$U^{+}\Gamma U = \gamma \tag{A1-2}$$

However, any eigenvector can still be multiplied in all its components by any complex number of unit magnitude, i. e., by a factor of the form exp ($i\theta$), without affecting the statement in Eq. A1-1. The corresponding effect on the construction of U is that U in Eq. A1-2 can be replaced by <u>any</u> unitary matrix, V, of the form

$$V = UD \tag{A1-3}$$

where D is a diagonal unitary matrix with the form

$$D = \begin{pmatrix} \exp(i\theta_1) & & & & 0 \\ & \exp(i\theta_2) & & & \\ & & \exp(i\theta_3) & & \\ & & & \ddots & \\ 0 & & & & \exp(i\theta_M) \end{pmatrix} \tag{A1-4}$$

and where the θ_1, θ_2. . . . θ_M are quite arbitrary. The matrix D clearly has the properties of being both unitary <u>and</u> symmetric,

$$D^T = D$$
$$D^{+}D = I \tag{A1-5}$$

where I is the identity matrix, whose elements are δ_{ij}.

Next, suppose Γ has a degenerate (multiple) eigenvalue. If the order of degeneracy (multiplicity) of a particular eigenvalue is r, then the secular equation determining the corresponding eigenvectors gives additional r-1 undetermined coefficients (i.e., in addition to the usual indeterminancy within a complex scale factor for the vector as a whole). One is then free to choose these coefficients to form r linearly independent eigenvectors (automatically orthogonal to the eigenvectors corresponding to any of the other eigenvalues), and in particular usually to construct them as a mutually orthogonal and normalized set (e.g., the Gram-Schmidt procedure) of r eigenvectors, which can then in turn be used in constructing U. However, given one such set, any other set of r linear combinations of these, constructed so as to provide orthonormality will also be suitable for constructing U. An orthonormality-preserving transformation from one subset to another is, of course, just an arbitrary unitary transformation in the subspace spanned by the eigenvectors of the particular eigenvalue. I.e., in the case of degenerate eigenvalues, we must extend Eq. A1-3 by the statement that more generally, given any U, we can equally well satisfy Eq. A1-2 by any

$$W = UD \qquad \text{(A1-6)}$$

where D has the quasi-diagonal <u>partitioned form</u>

$$D = \begin{pmatrix} D^{(1)} & & & & & 0 \\ & D^{(2)} & & & & \\ & & D^{(3)} & & & \\ & & & D^{(4)} & & \\ & & & & \ddots & \\ 0 & & & & & D^{(m)} \end{pmatrix} \qquad \text{(A1-7)}$$

The j^{th} partition, $D^{(j)}$, is <u>any</u> unitary matrix of order $r_j \times r_j$, where r_j is the order of degeneracy (multiplicity) of the j^{th} different eigenvalue of Γ, in some assumed ordering of the eigenvalues. In this form, the columns of $D^{(j)}$ include the arbitrary phase factors outlined in Eq. A1-4; in fact for any non-degenerate eigenvalue, the corresponding partition $D^{(j)}$ is only a single element of general form $\exp(i\theta_j)$.

It is to be noted that if the matrix γ is represented in a partitioned form similar to Eq. A1-7, the partitions are a set of identity matrices, multiplied by scalars,

$$\gamma = \begin{pmatrix} \boxed{\gamma_1 I} & & & & \\ & \boxed{\gamma_2 I} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot & \\ & & & & & \boxed{\gamma_N I} \end{pmatrix} \tag{A1-8}$$

and hence γ commutes with any D matrix of the form given in Eq. A1-7,

$$\gamma D = D\gamma \tag{A1-9}$$

APPENDIX 2: General Solution for S in Singular Case

If some of the λ_k in Eq. 7 are zero, then Λ^{-1} is strictly undefined, and we cannot proceed as in Eq. 11a. However, we can certainly still write Eq. 11b,

$$S = T\Lambda U^+$$

to introduce a new matrix T. If some of the diagonal elements of the diagonal matrix of Λ vanish, then it is readily observed that in the matrix multiplication of Eq. A2-1, the corresponding columns of T (whatever their values may be) never contribute to the elements of S; likewise the corresponding rows of U^+ (or columns of U) never enter. This is really the implication of the singularity in Λ, that the new unknown matrix T can never be completely specified from a knowledge of S. However, since S rather than T is our true objective, this is no hindrance to our solution.

Further, as in Eq. 13, we can also introduce a new vector ζ,

$$\zeta = \Lambda z$$

$$\zeta^+ = z^+\Lambda^+ \qquad \text{(A2-2)}$$

in which, by the property of some of the λ_k vanishing, the corresponding elements of ζ always vanish. Thus this ζ is arbitrary only in those elements ζ_k, corresponding to non-vanishing λ_k.

We thus again derive the inequality of Eq. 14.

$$\zeta^+\zeta \geq \zeta^+T^+T\zeta \qquad \text{(A2-3)}$$

But now, due to the zero elements just mentioned in the class of otherwise arbitrary ζ-vectors, this inequality (as is readily observed by inspection) in fact specifies limitations on all columns of T, except exactly those already mentioned as being irrelevant to the form of S. The matrix T^+T is always Hermitian, since $(T^+T)^+ = T^+T$, and positive definite since $Z^+T^+TZ = (TZ)^+(TZ) \geq 0$, so a representation as in Eq. 15 is always possible

$$T^+T = P^+\mu^2P \qquad \text{(A2-4)}$$

where P is again unitary, and μ^2 diagonal with elements positive real or zero. But now (inserting a step previously left to the reader between Eqs. 15a and 16), Eq. A2-3 becomes

$$\zeta^+\zeta \geq \zeta^+P^+\mu^2P\zeta \qquad \text{(A2-5)}$$

We can now introduce the transformation

$$\eta = P\zeta \qquad \text{(A2-6)}$$

where η can be any arbitrary vector provided that in the inverse transformation

$$\zeta = P^{+}\eta \tag{A2-7}$$

the required elements of ζ always vanish. This implies that for each arbitrary η, the appropriate rows of P^{+} (columns of P) must always be such as to produce this vanishing of the required elements of S. But, regarding this as a specification to be satisfied by any P providing a solution through Eq. A2-4, the inequality of Eq. A2-5 becomes just

$$\eta^{+}\eta \geq \eta^{+}\mu^{2}\eta \tag{A2-8}$$

Or, again,

$$\mu_k^2 \leq 1 \quad \text{for all k} \tag{A2-9}$$

We can certainly again factor Eq. A2-4, to give

$$T = W\mu P$$

where W is <u>any</u> unitary matrix; and thus from Eq. A2-1, write (exactly as in Eq. 19)

$$S = W\mu P\Lambda U^{+} \tag{A2-10}$$

Finally, we observe that the peculiar limitation on certain columns of P (especially peculiar-looking now since it is related to the form of an arbitrary vector, η, which does not enter our final equations) is in fact irrelevant since P enters only through the product $P\Lambda$, and it is exactly these columns of P which do not contribute to the computation of S because of the vanishing of the corresponding λ_k.

We have thus demonstrated our assertion that the result, Eq. 19, holds without change in the limiting case where Λ is singular through the vanishing of some of its diagonal elements.

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